

# An Exploration to Introduce the Formal System in Teaching Class

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## Abstract

Formal systems are a central issue in mathematical logic and mathematics. The student who has never encountered it before may feel that it is very abstract and difficult to understand. To overcome it, a more natural approach to introducing the formal system in teaching classes is presented. Also, this approach has been tested in the author's teaching class several times and received high evaluations. In this approach, several visual examples related to the mathematical logic and the formal system make the important effect on the understanding of the formal system. First, a simple and intuitive example about logic is presented, which can give students an initial feeling that the inference is not associated with the meaning of the sentences. Then, non-Euclidean geometry and Russell's paradox in set theory have enhanced this feeling in students again. After this groundwork, the introduction of the formal system becomes more understandable for students in the author's teaching class.

## Keywords

Formal system; set theory; non-Euclidean geometry; mathematical logic; meta-mathematics

## 1. Introduction

Formal system is a vital conception for the mathematical basis (Enderton, 2001; Hamilton, 1978; Kleene, 1971; Mendelson, 1979; Rautenberg, 2010), and it is always to be felt very abstract for the student who is the first time to encounter this conception. If a person starts learning the mathematical logic from scratch, of course, the difficulty of the formal system for this person to understand will be sharply reduced. In many cases, one doesn't have the mathematical logic background, and he or she just wants to learn the formal system from the historical and foundational perspective. So, how to let them not feel very complicated is a meaningful issue.

The author of this paper has been teaching a course on set theory at the university for a long time. The first three classes of his course are about the introduction to the development of the history of set theory, where the formal system is always discussed. However, the conception of the formal system is very abstract and thus difficult for a person who has never encountered it before.

In this paper, a more natural approach to introducing the formal system in the author's teaching class is proposed. Firstly, a very simple and intuitive example is designed, from which one can obtain a direct understanding of mathematical logic. Secondly, the focus of the person is distracted by non-Euclidean geometry. At first glance, one may feel it very strange for the distraction to non-Euclidean geometry, since it seems not related to the formal system. However, the person who is familiar with the famous work of "Foundation of Geometry" will feel that it is a better

way to understand David Hilbert's thinking of constructing a formal system for mathematics and logic simultaneously. Finally, the two understandable examples of the formal system are presented, and the person who has followed the previous two steps will not feel as difficult as before at this point.

## 2. A Simple and Intuitive Example of Logic

Logic is abstract and not easy to demonstrate to someone who has not learned logic before. If I say the logic is about the form of the sentence only and not associated with the meaning, you probably have no idea what I am talking about at all. In order to give an intuitive feeling about the logic for the students in the class, presenting some examples of the logic at first is necessary. Here, a simple and intuitive example of logic is designed as follows.

Inference (1): From the premises "Today is Monday or today is Tuesday" and "Today is not Monday", the conclusion "Today is Tuesday" can be obtained.

Inference (2): From the premises "This pen is black or this pen is blue" and "This pen is not black", the conclusion "This pen is blue" can be obtained.

Inference (3): From the premises "This natural number is odd or this natural number is even" and "This natural number is not odd", the conclusion "This natural number is even" can be obtained.

From the above three inferences, which have different meanings, a common property can be felt in some way. In fact, the common property is the common logic structure embedded in these three inferences, that is, their premises all talk with two different assertions and exclude the one, and thus the conclusion obtains the other one. If we use symbols to express the common logic structure, they can be expressed as follows.

Premises:  $p \text{ OR } q$   
 $\text{NOT } p$

Conclusion:  $q$

The symbols "p" and "q" are denoted as sentences that can be any assertions. In the above three inferences, the symbol p is the sentence as "Today is Monday", "This pen is black" or "This nature number is odd", and the symbol q is the sentence as "Today is Tuesday", "This pen is blue" or "This nature number is even". At this point, you can see that the inference does not depend on the meaning of the sentence in any way. It is the logical structure that determines the whole reasoning rather than the meaning of sentences (Geng & Qu, 1998; Hamilton, 1978; Mendelson, 1979). Even if you don't learn the logic beforehand, you can also get an intuition about the logic, which is about the inferences that are not associated with the meanings to some extent.

## 3. Non-Euclidean Geometry

Non-Euclidean geometry is a good demonstration to show that the inference is not associated with the meaning of the sentences. In the author's teaching class, a general introduction to non-Euclidean geometry is presented as follows.

In history, the monumental "Elements" by Euclid is the foundation of all mathematics (Kleene, 1971; Wang, 2001). From five axioms and five postulates, Euclid had proved more than four hundred geometric propositions. Among these ten axioms and postulates, the fifth postulate has attracted more attention. The fifth postulate is that "If a straight line intersects two straight lines and the sum of the two interior angles on the same side is less than two right angles, then the two lines will intersect on this side." This postulate is more complex than the other nine axioms and postulates, and thus, it is supposed that it can be derived from the other nine axioms and postulates. So, many people have tried to prove it, which has cost them a lot of time and energy. If it indeed can be proved from the other nine axioms and postulates, then after adding its negative form to the axioms and postulates, a pair of contradictory propositions will inevitably be obtained somewhere. However, along this idea, no contradictions have appeared, and on the contrary, many strange propositions can be derived. The equivalent form of the fifth postulate is the parallel postulate (Yue, 2020; Wang, 2001). That is, from a point that is out of a straight line, there is one and just one straight line that is parallel to the former straight line. Thus, the negative form of the parallel postulate may be that there is no such straight line that is parallel to the former straight line, or there is more than one straight line that is parallel to the former straight line. The different negative forms of the parallel postulate will derive different

non-Euclidean geometries, where one is called Riemannian geometry, and the other is called Lobachevskian geometry.

No contradictions are found in non-Euclidean geometry, but this does not mean that it does not contain contradictions. So, how can we handle this issue? The former intuitive feeling about the logic will have an effect on this issue. From Section I, we know that the logic is the inner structure of the inference. If a contradiction is embedded in non-Euclidean geometry somewhere that is a proposition and its negative is also to be derived in non-Euclidean geometry, then the contradiction will also appear in new meanings or explanations on non-Euclidean geometry since the logic is not associated with the meaning of the sentences (Yue, 2020; Wang, 2001). For example, we take the assertion as the postulate that from a point that is outside a straight line, there is no straight line that is parallel to the former straight line. If new interpretations or new meanings are given to terms in non-Euclidean geometry, then the propositions in non-Euclidean geometry will become the propositions in Euclidean geometry. Suppose that the “point” in non-Euclidean geometry is interpreted as a pair of vertices on a sphere in Euclidean geometry, the “line” in non-Euclidean geometry is interpreted as the great circle on a sphere in Euclidean geometry, and the “plane” in non-Euclidean geometry is interpreted as the surface of the sphere in Euclidean geometry (Yue, 2020; Wang, 2001). Under this new interpretation, the proposition that a straight line can be drawn through two points becomes a great circle that can be drawn through two pairs of vertices on the sphere, and the proposition that from a point that is out of a straight line there is no straight line parallel to the former straight line becomes any two great circles will intersect on the sphere. In Figure 1, these two propositions are shown schematically for understanding. We can see that if there is a contradiction in non-Euclidean geometry, then there inevitably is a contradiction in Euclidean geometry.

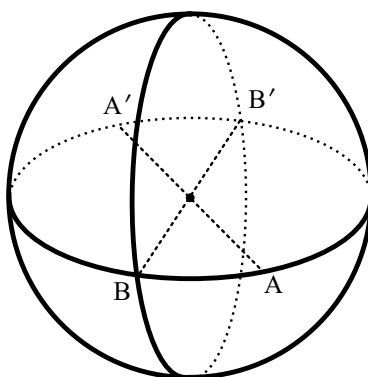


Figure 1. The non-Euclidean geometry is shown schematically.

What's next is how to show there is no contradiction in Euclidean geometry. In David Hilbert's "Fundamentals of Geometry", the issue of whether there is a contradiction in Euclidean geometry is transformed into the issue of whether there is a contradiction in the theory of real numbers. The method used here is based on the coordinate system proposed by Descartes (Yue, 2020; Wang, 2001). Suppose that the “point” in Euclidean geometry is interpreted as a pair of real numbers that is denoted as  $(x, y)$ , the “line” in Euclidean geometry is interpreted as the ratio of three real numbers that is denoted as  $a:b:c$ . As the propositions in non-Euclidean geometry will become the propositions in Euclidean geometry if new interpretations or new meanings are given, the propositions in Euclidean geometry will become the propositions in the theory of real numbers under this new interpretation. For example, in Euclidean geometry, the proposition that two points determine one straight line becomes the proposition

$$a \times x_1 + b \times y_1 + c = 0 \text{ and } a \times x_2 + b \times y_2 + c = 0,$$

where  $(x_1, y_1)$  and  $(x_2, y_2)$  are two points. Since  $c = -(a \times x_1 + b \times y_1) = -(a \times x_2 + b \times y_2)$ , we get

$$a:b = -(y_1 - y_2):(x_1 - x_2)$$

Multiplying  $x_2$  to  $a \times x_1 + b \times y_1 + c = 0$ , and  $x_1$  to  $a \times x_2 + b \times y_2 + c = 0$ , we have

$$a \times x_1 \times x_2 + b \times y_1 \times x_2 + c \times x_2 = 0, \text{ and } a \times x_2 \times x_1 + b \times y_2 \times x_1 + c \times x_1 = 0.$$

Subtracting these two equations, we have

$$b \times (y_1 \times x_2 - x_1 \times y_2) + c \times (x_2 - x_1) = 0.$$

Therefore, we have

$$b : c = -(x_2 - x_1) : (y_1 \times x_2 - x_1 \times y_2).$$

Now, we can see that the only line determined is  $a : b : c = -(y_1 - y_2) : (x_1 - x_2) : (y_1 \times x_2 - x_1 \times y_2)$ .

From this example, we can see that this proposition in Euclidean geometry is transformed into a proposition expressed by the theory of real numbers. Due to the fact that the propositions in Euclidean geometry can be transformed into propositions in the theory of real numbers, if there is a contradiction in Euclidean geometry, then there will inevitably be a contradiction in the theory of real numbers.

For the real number, it can be reconstructed by the rational number, in which a Cauchy rational sequence  $\{q_n\}$  is considered to be a real number. Similarly, we can use a pair of integer numbers  $(k, l)$  to express a rational number, and a pair of natural numbers  $(m, n)$  to express an integer number. This process is the standard method to reconstruct the number system. So, the issue of whether there is a contradiction is reduced to the issue of whether there is a contradiction in the theory of natural numbers.

In this part, Euclidean geometry is very familiar to the students in the teaching class, whereas non-Euclidean geometry is not. The key idea in this part is that non-Euclidean geometry can be regarded as mathematics with respect to logic rather than the physical meanings of the space. So, we can give any interpretations to the meanings of objects in non-Euclidean geometry. This idea must be emphasized in the teaching class, and let the students pay more attention to this.

When teaching this part in the author's class, the reconstruction details of the number system can be presented to some degree. For example, some reconstruction details of real numbers and integer numbers from rational numbers and natural numbers can be presented respectively.

#### 4. Set Theory

The emergence of the paradox in set theory has led to research in mathematical logic, including the formal system. Some knowledge about the paradox in set theory is introduced in the teaching class.

The set theory was proposed by Georg Cantor in the late nineteenth century, and can be regarded as a foundation of the whole mathematics. Of course, we can derive the natural numbers in set theory. However, whether there is a contradiction in the set theory is the key point. The set theory proposed by Cantor is called the naïve set theory, where the conception of the set is imagined by our intuition. Here, a set is a collection of elements satisfying a certain property, which implies the comprehensive principle (Rautenberg, 2010; Wang, 2001). That is, if  $P(x)$  is about a property of the element  $x$ , then there is a set denoted by  $\{x | P(x)\}$ .

In 1902, Bertrand Russell found a contradiction or a paradox in set theory. That is, if the  $P(x)$  is  $x \notin x$ , then for the set  $y = \{x | x \notin x\}$ , there is a paradox when you consider whether or not  $y$  belongs to  $y$ . If  $y \notin y$ , then  $y$  will have the property  $P(x)$  that it doesn't belong to itself, so we have  $y \in y$ . On the other hand, if  $y \in y$ , then  $y$  has the property  $P(x)$  that it doesn't belong to itself, so we have  $y \notin y$ . We always obtain a proposition and its negative form in both cases, which means that a contradiction has appeared. The Russell paradox introduces a big shake in mathematics. Because the method of using a property to form a set is so usual in mathematics, it is consistent with our intuition. The Russell's paradox shows that our intuitive doesn't always right, and there may be other contradictions we have not encountered yet.

In fact, Russell's paradox is a logic paradox; that is to say, we don't need to know the meaning of the set (Rautenberg, 2010). The comprehensive principle can be formalized as  $\exists y \forall x (x \in y \leftrightarrow P(x))$ , where the variables  $x$  and  $y$  are denoted by sets by default. Then, the Russell's paradox is formalized as  $\exists y \forall x (x \in y \leftrightarrow x \notin x)$ . However,  $\forall x (x \in y \leftrightarrow x \notin x) \rightarrow (y \in y \leftrightarrow y \notin y)$  is logically valid, that is to say, it always holds whatever you interpret the variables  $x$  and  $y$ , and the meaning of the symbol  $\in$ . The consequence of this implication, i.e.,  $(y \in y \leftrightarrow y \notin y)$ , is a contradiction

under any meaning. So, the antecedent of this implication, i.e.,  $\forall x(x \in y \leftrightarrow x \notin x)$ , is also a contradiction under any meaning. Therefore, Russell's paradox can be understood as not related to set theory at all.

Mainly, this part is not difficult for the students in the teaching class. However, some basic conceptions in the first-order logic are not easy for them to understand. This demands that teachers give more intuitive examples in the teaching class.

## 5. Formal System

The set is the most fundamental object in mathematics, and the contradiction is the issue in logic. In order to overcome the contradiction in set theory, we should focus our effort to mathematic and logic simultaneously. Meanwhile, the method used in Section III to ensure a mathematical theory has no contradiction depends on another mathematical theory. In 1922, David Hilbert proposed his program, where he wanted to give a proof to show a mathematical theory has no contradiction (Yue, 2020; Wang, 2001). In his program, the proof itself in a mathematical theory becomes the mathematical object, and this proof is embedded into a formal system. The formal system is the kernel conception of his program. This idea comes from his book "Fundamentals of Geometry", where the meanings of geometric objects such as points and straight lines have been extracted to some degree. The formal system is a system that is completely formalized, where only the formal objects are in the system, such as symbols, strings of symbols, and sequences of strings of symbols. No meanings are put on symbols in the formal system.

The formal system contains four parts (Geng & Qu, 1998; Hamilton, 1978; Wang, 2001): 1. An alphabet of symbols, 2. A set of strings with a finite length of symbols, which is called the well-formed formulas, 3. A set of axioms that are selected from the well-formed formulas, 4. A set of inference rules that can enable one to obtain a well-formed formula from some other well-formed formulas. Through the formal system, the concept of metamathematics emerged. In the formal system, object comprised by symbols has no any meanings, and out of the formal system, different meanings can be imposed to these objects in the formal system. Conventional mathematics and metamathematics stay at different levels.

Of course, the conception of a formal system is very abstract to the person who is encountering it for the first time. Two examples of formal systems are given to help overcome this (Geng & Qu, 1998; Hofstadter, 1996).

Formal system 1:

Alphabet: p, q, #.

Formulas: A string with the form  $xqypz$ , where x, y, and z are finite strings of #.

Axioms:  $x\#qxp\#$ , where x is a finite string of #.

Rules: From string  $xqypz$ , the string  $x\#qypz\#$  can be obtained, where x, y, and z are finite strings of #.

For this formal system, there are only three symbols, i.e., p, q, and #. For the symbols such as x, y, and z, they are the symbols in metalanguage that denote the symbol # in object language. For example, according to the definition of the Formulas, the string  $###q###p\#$  is a formula, where x, y, and z are  $###$ ,  $##$ , and  $\#$ , respectively. The axiom is given in the form of the axiom scheme, which expresses many specific axioms. For example, if x is #, then  $##q\#p\#$  is an axiom; and if x is ##, then  $###q###p\#$  is also an axiom. According to the axioms and rules in the formal system, we can obtain many theorems. For example, we say that  $#####q\#p###$  is a theorem. The proof is shown as follows.

- (1)  $##q\#p\#$  According to the axiom where x is #.
- (2)  $###q\#p##$  Use the rule to (1) where x is ##, y is #, and z is #.
- (3)  $#####q\#p###$  Use the rule to (2) where x is ###, y is #, and z is ##.

From the proof, we can see that the whole proving process is just the transformation of the symbols in the formal system without any meaning. However, from the outside of the formal system, we can impose meanings on these symbols. For example, we let q be the meaning of "equal", p to be the meaning of "plus", # to be the meaning of the natural number 1, ## to be the meaning of the natural number 2, and so on. Under this interpretation, the meaning of axiom is  $(n+1)=n+1$ , the meaning of rule is from  $n=m+k$  we can obtain  $(n+1)=m+(k+1)$ , and the theorem is  $4=1+3$ . Obviously, we can have another interpretation where q has the meaning of "minus", p has the meaning of "equal", # to be the meaning of the natural number 1, ## to be the meaning of the natural number 2, and so on. Under this interpretation, the meaning of axiom is  $(n+1)-n=1$ , the meaning of rule is from  $n-m=k$  we can obtain  $(n+1)-m=(k+1)$ ,

and the theorem is  $4-1=3$ .

When you see the formal system 1 for the first time, you are probably feeling unbelieving that the mathematical meanings of plus and minus can be expressed as this. The whole proving process is completely just the transformation of the symbols, which is wholly mechanical without any understanding of these symbols.

Formal system 1 is not associated with the logic, whereas the second formal system has relations to the logic. For the second one, we will give a brief introduction to it. We use symbols  $\neg$ ,  $\wedge$ ,  $\vee$  and  $\rightarrow$  to express the meanings of “not”, “and”, “or” and “if ..., then ...”. We also use the symbols  $p$ ,  $q$ ,  $r$ , and  $s$  to express the meanings of statements. Then, from our intuition, we have the following three logical laws: (I) From  $p \wedge q$ , we can obtain  $p$  and  $q$ ; (II) From  $p$  and  $p \rightarrow q$ , we can obtain  $q$ ; (III) From  $p \vee q$  and  $\neg p$ , we can obtain  $q$ . Based on these logical laws, we have the following inference.

Premises:  $p \rightarrow (q \vee r)$ ,  $(\neg s) \rightarrow (\neg q)$ ,  $p \wedge (\neg s)$

Conclusion:  $r$

The inference process is

- |     |                                 |   |
|-----|---------------------------------|---|
| (1) | $p \wedge (\neg s)$             | According to the premise.               |
| (2) | $p$                             | Use the logic law (I) to (1).           |
| (3) | $\neg s$                        | Use the logic law (I) to (1).           |
| (4) | $(\neg s) \rightarrow (\neg q)$ | According to the premise.               |
| (5) | $\neg q$                        | Use the logic law (II) to (3) and (4).  |
| (6) | $p \rightarrow (q \vee r)$      | According to the premise.               |
| (7) | $q \vee r$                      | Use the logic law (II) to (2) and (6).  |
| (8) | $r$                             | Use the logic law (III) to (5) and (7). |

Like the formal system 1, the whole inference process in formal system 2 is completely the transformation of the symbols, which is wholly mechanical without any understanding of these symbols. We can impose meanings on these symbols from the outside of the formal system 2. First, we let  $p$  to be the meaning of “ $x$  is a real number”,  $q$  to be the meaning of “ $x$  is a rational number”,  $r$  to be the meaning of “ $x$  is an irrational number”, and  $s$  to be the meaning of “ $x$  can be expressed as a fraction”. In this interpretation, the premise of the inference is “ $x$  is a real number implies  $x$  is either a rational number or an irrational number,  $x$  cannot be expressed as a fraction implies  $x$  is not a rational number,  $x$  is a real number and a cannot be expressed as a fraction.” While the conclusion of the inference is “ $x$  is an irrational number.” Of course, we can impose the second meaning on this inference. Suppose the students in the classroom now come from class 1 and class 2, and the surname of the head teacher of class 1 is Bai. So, we let  $p$  to be the meaning of “student  $x$  is in the classroom”,  $q$  to be the meaning of “student  $x$  comes from class 1”,  $r$  to be the meaning of “student  $x$  comes from class 2”, and  $s$  to be the meaning of “the surname of the head teacher of student  $x$  is Bai”. In this interpretation, the premise of the inference is “student  $x$  is in the classroom implies student  $x$  either comes from class 1 or comes from class 2, the surname of the head teacher of the student  $x$  is not Bai implies the student  $x$  is not comes from class 1, student  $x$  is in the classroom and the surname of the head teacher of the student  $x$  is not Bai.” While the conclusion of the inference now becomes “student  $x$  comes from class 2.” Here we feel what the logic is once again. From these two simple examples of the formal system, even if it is the first time for the person to encounter the formal system, he or she can also get an intuitive feeling of it. We can imagine that in the formal system by David Hilbert, the logic and mathematics are both formalized into it, and the meaning can be obtained from the view of metamathematics.

The author of this paper has conducted experiments in his teaching class several times. By the process listed above, the feedback from the students in the course is very good. Some students find it is too surprising that mathematics can be understood like that.

## 6. Conclusions

In this paper, the author has tried to present a more natural method to introduce the concept of a formal system, which is very abstract and difficult for a person who has never encountered it before. The key point is to understand that the logic is not associated with the meaning of the inference, and it is just about the form of inference. Some

simple and visualized examples are designed to demonstrate this key point. Also, these examples are presented step by step and in accordance with the historical occurrences.

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