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Almost Copositive Tensors and Related Research

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Abstract

This paper extends the theory of almost copositive matrices to higher-order symmetric tensors. For even-order real symmetric tensors, we first introduce definitions for almost copositive tensors, almost copositive-plus tensors, and the concept of a key index. These definitions provide a unified framework for analyzing copositivity-related properties in tensor spaces. Subsequently, by formulating a constrained optimization model and applying the Karush-Kuhn-Tucker (KKT) conditions, we derive properties of the eigenvalues of almost copositive tensors. Finally, we prove the existence of key indices for almost copositive-plus tensors and demonstrate that their number is at most two. Overall, this work generalizes important matrix results to higher-order tensors and provides new theoretical tools for future studies in tensor optimization and related applications.

Keywords

Almost copositive tensor; copositive-plus tensor; symmetric tensor; eigenvalue

1. Introduction

Copositive matrices play an important role in quadratic programming, combinatorial optimization, and machine learning. A matrix A is said to be copositive if $x^TAx \geqslant 0$ for any nonzero nonnegative vector x. Furthermore, if Ax = 0 whenever $x^TAx = 0$, then A is called a copositive-plus matrix; if x = 0 whenever $x^TAx = 0$, then A is said to be strictly copositive. In recent years, with the emergence of high-dimensional data problems, tensors as a generalization of matrices have gradually attracted researchers' attention [1]. In 2013, Qi [2, 3] extended copositivity to tensors and provided relevant properties of copositive tensors. Subsequently, many researchers have investigated copositive tensors in physics, hypergraph theory, polynomial optimization, and other fields [4-7]. In matrix theory, an nth-order matrix A is said to be almost copositive if it is not copositive, but every one of its (n−1) th-order principal submatrices is copositive in [1].

Definition 1.1 [1]

A symmetric matrix $A^T \in R^{n\times n}$ is said to be k-order copositive (copositive plus, strictly copositive) if every principal submatrix of A of order k is copositive (copositive plus, strictly copositive). Matrix A is said to be exactly k-order copositive (respectively, copositive-plus, etc.) if it is copositive (copositive-plus, etc.) of order k but not of order k+1.

Definition 1.2 [1]

Matrix $A = A^T \in R^{n\times n}$ is called an almost copositive(-plus) matrix if it is exactly of order n−1 copositive(-plus).

Building upon the literature [1], this paper extends the concept of almost copositivity to higher-order tensors. We first provide precise definitions for copositive-plus tensors, etc., while introducing the notion of key indices to characterize almost copositive-plus tensors. Subsequently, by constructing a constrained polynomial optimization model, we prove that an almost copositive tensor necessarily possesses a negative eigenvalue corresponding to a positive eigenvector. Finally, we demonstrate that key indices exist for almost copositive-plus tensors and that there are at most two such indices. These results not only enrich the theory of tensor optimization but also provide new tools for further research on tensor eigenvalue distributions and polynomial optimization problems.

2. Main Results

In this paper, we denote the n-dimensional Euclidean space by R^n. Scalars are represented by lowercase letters a, b, c, x, ⋯, while vectors in R^n are denoted by bold letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}, \ldots$ The set of all mth-order n-dimensional symmetric tensors is denoted by $S_{m,n}$. The notation $\mathbf{x} \geq 0$ indicates that all components of the vector x are nonnegative. The ith component of vector x is denoted by $\mathbf{x}$. For any tensor $\mathcal{A}$ and vector $\mathrm{x} \in \mathrm{R}^n$, we define

$$\mathcal{A}\mathrm{x}^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1, i_2, \ldots, i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

$$(\mathcal{A}\mathrm{x}^{m-1})_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i, i_2, \ldots, i_m} x_{i_2} \cdots x_{i_m.}$$

This paper considers real symmetric tensors $\mathcal{A} \in \mathrm{S}_{m,n}$ of even order m so that real eigenvalues exist. For a non-symmetric tensor $\mathcal{B}$. We consider its symmetrization $Sym\mathcal{B}$. Since $\mathcal{B}\mathrm{x}^m = \mathrm{Sym}\mathcal{B}\mathrm{x}^m$, the copositivity of $\mathcal{B}$ is equivalent to that of its symmetrized version. Hence, without loss of generality, we restrict our attention to symmetric tensors.

Definition 2.1 Let $\mathcal{A} \in S_{m,n}$ be a copositive tensor. If for every $\mathbf{x} \in R_+^n$ satisfying $\mathcal{A}\mathrm{x}^m = 0$, we have $\nabla(\mathcal{A}\mathbf{x}^m) = 0$, then $\mathcal{A}$ is called a copositive-plus tensor.

Definition 2.2 A tensor $\mathcal{A}$ is said to be almost copositive if it is not copositive itself, but all its (n-1)-dimensional principal subtensors are copositive.

Definition 2.3 A tensor $\mathcal{A}$ is called almost copositive-plus if it is copositive but not copositive-plus, while all its (n-1)-dimensional principal subtensors are copositive-plus.

Definition 2.4 A tensor $\mathcal{A}$ is called almost strictly copositive if it is copositive but not strictly copositive, while all its (n-1)-dimensional principal subtensors are strictly copositive.

Definition 2.5 Let $\mathcal{A}$ be an almost copositive-plus tensor. If its principal subtensor $\mathcal{A}_S$ with $S = N \setminus \{k\}$ is almost strictly copositive, then the index $k \in \{1,2,\ldots,n\}$ is called a key index of $\mathcal{A}$.

Theorem 2.1 Let $\mathcal{A}$ be an m-th order n-dimensional almost copositive tensor. Then

(1) $\mathcal{A}$ possesses at least one negative eigenvalue;

(2) There exists a positive eigenvector corresponding to a negative eigenvalue.

Proof: (1) Consider the optimization problem:

(P): $min\{\mathcal{A}x^m: x \in R_+^n, ||x||_m = 1\}$,

where $||\boldsymbol{x}||_m = (\sum_{i=1}^n |x_i|^m)^{1/m}$. Since the feasible set is closed and bounded, it is compact, and the function $\mathcal{A}\boldsymbol{x}^m$ is continuous. Hence, the optimization problem attains a minimum. Denote this minimum by $\lambda_{\min} = \min \mathcal{A}\,\boldsymbol{x}^m$.

Because $\mathcal{A}$ is almost copositive, it is not copositive. Thus, there exists $\mathrm{y} \in \mathrm{R}_+^n$ such that $\mathcal{A}y^m < 0$. Let $\bar{y} = \frac{y}{||y||_m}$, then $\bar{y}$ is feasible for problem (P). Moreover,

$$\mathcal{A}\overline{y^m} = \mathcal{A}\left(\frac{y}{||y||_m}\right)^m = \frac{1}{||y||_m^m}\mathcal{A}y^m < 0$$

Consequently,

$\lambda_{min} \leq \mathcal{A}\overline{y^m} < 0$.

Hence $\mathcal{A}$ possesses at least one negative eigenvalue.

(1) Let $\boldsymbol{x}^*$be an optimal solution of the optimization problem (P). We now show $x^* > 0$.

Suppose $\exists i \in R, s.t. x_i^* = 0$. Denote $x_{N-i}^* = (x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_n^*)^\top \in R_+^{n-\mathbb{1}}$. We have

$$\mathcal{A}x^m = \sum_{i_1, i_2, \ldots, i_m \in N\setminus\{i\}} a_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m} + \sum_{\exists\, i_1, i_2, \ldots, i_m = i} a_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m} = \mathcal{A}_{\mathcal{N}-i}(x_{N-i}^*)^m.$$

Thus, $\mathcal{A}(x^*)^m = \mathcal{A}_{\mathcal{N}-i}(x^*_{N-i})^m$. Because $\lambda_{min} = \mathcal{A}(x^*)^m < 0$. we obtain $\mathcal{A}_{\mathcal{N}-i}(x^*_{N-i})^m < 0$, which contradicts the assumption that $\mathcal{A}$ is almost copositive. Hence $x^* > 0$.

Next, we prove that x^* is an eigenvector of $\mathcal{A}$.

Write problem (P) in standard form:

$$\min_{x \in R^n} f(x) = \mathcal{A}x^m$$

$$\text{s.t.}\, g(x) = \left\|x\right\|_m^m - 1 = 0$$

$$h_i(x) = -x_i \le 0, i = 1, \dots, n.$$

Clearly, when $x^* > 0$ the only active constraint is $g(x) = 0$. Therefore, the Lagrange function is
$L(x, \mu) = f(x) - \mu g(x) = \mathcal{A}x^m - \mu(||x||_m^m - 1), \mu \in R$,
where μ is the multiplier for the equality constraint.

Moreover,
$\nabla g(x^*) = \nabla(\sum_{i=1}^n (x_i^*)^m - 1) = m((x_1^*)^{m-1}, (x_2^*)^{m-1}, \dots, (x_n^*)^{m-1})^\top = m(x^*)^{[m-1]} > 0$.

Hence $\nabla g(x^*)$ is linearly independent, and there certainly exists a nonzero vector d such that $\nabla g(x^*)^\top d < 0$. Consequently, the MFCQ holds, and the KKT conditions are satisfied.

$\nabla L(x^*, \mu) = \nabla \mathcal{A}(x^*)^m - \mu \nabla(||x||_m^m - 1)|_{x=x^*} = 0$,

$m\mathcal{A}(x^*)^{m-1} = \mu m(x^*)^{[m-1]}$,

$$\mathcal{A}(x^*)^{m-1} = \mu(x^*)^{[m-1]}.$$

Multiplying both sides of the last equality by $(x^*)^\top$ yields

Right-hand side: μ,

Left-hand side: $(x^*)^\top \mathcal{A}(x^*)^{m-1} = x^*\left(\frac{1}{m}\nabla\mathcal{A}(x^*)^m\right) = \frac{1}{m}(x^*\nabla\mathcal{A}(x^*)^m) = \frac{1}{m}\cdot m\mathcal{A}(x^*)^m = \mathcal{A}(x^*)^m$.

Therefore $\mathcal{A}(x^*)^m = \mu = \lambda_{min} < 0$. Consequently, μ is a negative eigenvalue of $\mathcal{A}$ and the corresponding eigenvector x^* is positive.

Lemma 2.1 Let $\mathcal{A}$ be an almost copositive-plus tensor, and let x be a nonzero nonnegative vector such that $\mathcal{A}x^m = 0$ but $\nabla\mathcal{A}x^m \ne 0$. Then the vector x has exactly one zero component.

Proof: Let $K = \{i \in \{1, \dots, n\}: x_i = 0\}$. We shall show that $|K| = 1$.

Suppose $|K| = \emptyset$, i.e., $x > 0$, which obviously means x is an interior point and a global minimum point. By Fermat theorem, we then have $\nabla\mathcal{A}x^m = 0$, which contradicts the hypothesis that $\mathcal{A}$ is not copositive-plus. Hence, $|K| \ne \emptyset$.

Assume that $|K| \ge 2$. For every $i \in K$, define the sub-vector $\tilde{x} = x_{N-i} \in R_+^{n-1}$.

$$\mathcal{A}\boldsymbol{x}^{\boldsymbol{m}} = \sum_{i_1, i_2, \dots, i_m \in N\setminus\{i\}} a_{i_1, i_2, \dots, i_m} x_{i_1} x_{i_2} \cdots x_{i_m} + \sum_{\exists\, i_1, i_2, \dots, i_m = i} a_{i_1, i_2, \dots, i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$
$$= \sum_{i_1, i_2, \dots, i_m \in N\setminus\{i\}} a_{i_1, i_2, \dots, i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$
$$= \mathcal{A}_{\mathcal{N}-i}\widetilde{x^m} = 0.$$

Thus, $\mathcal{A}_{\mathcal{N}-i}\widetilde{x^m} = \mathcal{A}x^m$, and consequently $\nabla\mathcal{A}_{\mathcal{N}-i}\widetilde{x^m} = (\nabla\mathcal{A}x^m)_{N-i}$.

Because $\mathcal{A}$ is almost copositive-plus, by definition $\mathcal{A}_{\mathcal{N}-i}$ is copositive-plus. Hence, $\nabla\mathcal{A}_{\mathcal{N}-i}\widetilde{x^m} = (\nabla\mathcal{A}x^m)_{N-i} = 0$, so $(\nabla\mathcal{A}x^m)_k = 0\ \forall k \ne i$.

Since $|K| \ge 2$, there exists $j \in K$ with $j \ne i$ and $\tilde{x}_j = 0$. By the same argument, we also obtain $(\nabla\mathcal{A}x^m)_k = 0$ for all $k \ne j$. Therefore $\nabla\mathcal{A}x^m = 0$, which contradicts the definition of an almost copositive-plus tensor.

Consequently, |K| = 1.

Theorem 2.2 Let $\mathcal{A}$ be an almost copositive-plus tensor. Then $\mathcal{A}$ possesses a key index.

Proof: Since $\mathcal{A}$ is almost copositive-plus, Lemma 2.1 guarantees $\exists x^* \ge 0, s.t. \mathcal{A}(x^*)^m = 0, \nabla\mathcal{A}(x^*)^m \ne 0$

Let $K = \{k\}$, i.e., $x_k^* = 0$ and $x_i^* > 0 \forall i \ne k$. Denote $S = N \setminus \{k\}$, then $x_S^* > 0$.

(1) Because $\mathcal{A}$ is almost copositive-plus, hence $\mathcal{A}_S$ is copositive.

(2) Since $x_S^* \in R_+^{n-1}$ and $x_S^* > 0, \mathcal{A}_S(x_S^*)^m = \mathcal{A}(x^*)^m = 0$. Therefore, $\mathcal{A}_S$ is not strictly copositive.

(3) Assume that an (n-2)-dimensional principal subtensor of $\mathcal{A}_S$ is not strictly copositive. Then, there exists $j \in S$ such that $\mathcal{A}_R$ is not strictly copositive, where $R = N \setminus \{k,j\} = S \setminus \{j\}$. So, $\exists y \geq 0, s.t. \mathcal{A}_{\mathcal{R}} y^m = 0$.

Define $z \in R_+^n$ by $z_k = z_j = 0$ and fill the remaining components with y. Then $\mathcal{A}z^m = \mathcal{A}_{\mathcal{R}} y^m = 0$. Also, we have $\mathcal{A}_S z_S^m = 0$ (here z_S satisfies $z_j = 0$ and the other components are given by y). Because $\mathcal{A}_S$ is copositive-plus, it follows that $\nabla \mathcal{A}_S z_S^m = 0$. Similarly, $\nabla \mathcal{A}_{\mathcal{N}-j} z_{N-j}^m = 0$. From these two equalities and the fact that $i \neq j$, we obtain $\nabla \mathcal{A} z^m = 0$.

Let $w(t) = (1-t)z + tx^*), (t \in [0,1])$. For sufficiently small t > 0, we have $w(t) \geq 0$.

By the Taylor formula, we have

$\mathcal{A}w(t)^m = \mathcal{A}z^m + t \cdot m(x^* - z)^\top \nabla \mathcal{A} z^m + \frac{t^2}{2} Q(x^* - z) + O(t^3) = \frac{t^2}{2} Q(x^* - z) + O(t^3)$.

1) If $Q(x^* - z) < 0$, there exists $t_1 > 0$ such that $\mathcal{A}w(t_1)^m < 0$, contradicting the copositivity of $\mathcal{A}$.

2) If $Q(x^* - z) \geq 0$, then for sufficiently small $t > 0$ we have $\mathcal{A}w(t)^m \geq 0$. Set $d = z - x^*$.

If $d^\top \nabla \mathcal{A}(x^*)^m \leq 0$, the function value decreases along the direction d from x^*; hence there exists $t_2 > 0$ such that $\mathcal{A}(x^* + t_2 d)^m < 0$, a contradiction.

If $d^\top \nabla \mathcal{A}(x^*)^m > 0$, the gradient at x^* along $-d$ is negative, so the function value decreases along $-d$; therefore, there exists $t_3 > 0$ such that $\mathcal{A}(x^* - t_3 d)^m < 0$, again a contradiction.

$\nabla \mathcal{A}_S (x_S^*)^m = (\nabla \mathcal{A}(x^*)^m)_S$.

Since $\mathcal{A}$ is an almost copositive-plus tensor, we have $\nabla \mathcal{A}(x^*)^m \neq 0$.

Consequently, all (n-2)-dimensional principal subtensors of $\mathcal{A}_S$ are strictly copositive.

To summarize, the index K is a key index of $\mathcal{A}$.

Lemma 2.2 Let $\mathcal{B}$ be an m-th order n-dimensional almost strictly copositive tensor. If there exists a nonzero vector $y \in R_+^n$ such that $\mathcal{B}y^m = 0$, then necessarily $y > 0 \; \forall i \in \{1, \ldots, n\backslash\}$.

Proof: Suppose that there exists an index j with $y_j = 0$. Let $S = \{1, \ldots, n\} \setminus \{j\}$ and consider the principal subtensor $\mathcal{B}_S$.

Since $y_j = 0$, we have $\mathcal{B}y^m = \mathcal{B}_S(\tilde{y})^m$, where $\tilde{y}$ is the vector obtained from y by deleting the j-th component. By hypothesis $\mathcal{B}_S$ is strictly copositive, and $\tilde{y} \geq 0$ is nonzero; hence $\mathcal{B}_S(\tilde{y})^m > 0$. But the left-hand side $\mathcal{B}y^m$ equals 0, a contradiction.

Therefore, $y > 0$.

Theorem 2.3 An almost copositive-plus tensor $\mathcal{A}$ has at most two key indices.

Proof: Assume that $k_1, k_2, k_3 \in \{1, \ldots, n\}$ are three distinct key indices of $\mathcal{A}$. For each k_r the principal subtensor $\mathcal{A}_{S_r}$ with $S_r = N \setminus \{k_r\}$ is almost strictly copositive. Hence there exists a nonzero vector $y^{(r)} \in R_+^{n-\mathbb{1}}$ such that $\mathcal{A}_{S_r}\left(y^{(r)}\right)^m = 0$. By Lemma 2.2, we have $y^{(r)} > 0$.

Define $x^{(r)} \in R_+^n$ by $x_{k_r}^{(r)} = 0$ and $x_i^{(r)} = y_i^{(r)} > 0 \; (i \neq k_r)$. Then

$$\mathcal{A}\left(x^{(r)}\right)^m = \mathcal{A}_{S_r}\left(y^{(r)}\right)^m = 0 (r = 1,2,3). \tag{1}$$

Moreover, since $\mathcal{A}_{S_r}$ is copositive-plus, we obtain$\nabla \mathcal{A}_{S_r}\left(y^{(r)}\right)^m = 0$ i.e.,

$$\left(\nabla \mathcal{A}\left(x^{(r)}\right)^m\right)_j = 0, \forall j \neq k_r. \tag{2}$$

Denote $g_r = \left(\nabla \mathcal{A}\left(x^{(r)}\right)^m\right)_{k_r}$. Then $\nabla \mathcal{A}\left(x^{(r)}\right)^m = g_r e_{k_r}$, where e_{k_r} is the unit vector with the k_r-th component equal to 1 and the others 0.

Let $u = \alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \alpha_3 x^{(3)}$with $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Expanding $\mathcal{A}u^m$ gives

$\mathcal{A}u^m = \mathcal{A}\left(x^{(1)}\right)^m + m\left(u - x^{(1)}\right)^\top \nabla \mathcal{A}\left(x^{(1)}\right)^m + \frac{1}{2} Q_1(\alpha)$,

where $Q_1(\alpha)$ is the second-order remainder. Substituting equation (1) and equation (2) yields

$$\mathcal{A}u^m = m\left(\alpha_2 x_{k_1}^{(2)} + \alpha_3 x_{k_1}^{(3)}\right) g_1 + \frac{1}{2} Q_1(\alpha). \tag{3}$$

Similarly, we obtain

$$\mathcal{A}u^m = m\left(\alpha_1 x_{k_2}^{(1)} + \alpha_3 x_{k_2}^{(3)}\right) g_2 + \frac{1}{2} Q_2(\alpha), \tag{4}$$

$$\mathcal{A}u^m = m\left(\alpha_1 x_{k_3}^{(1)} + \alpha_2 x_{k_3}^{(2)}\right) g_3 + \frac{1}{2} Q_3(\alpha). \tag{5}$$

Since the left-hand sides are identical, the linear terms coincide:

$$\left[\alpha_2 x_{k_1}^{(2)} + \alpha_3 x_{k_1}^{(3)}\right] g_1 = \left[\alpha_1 x_{k_2}^{(1)} + \alpha_3 x_{k_2}^{(3)}\right] g_2 + \left[\alpha_1 x_{k_3}^{(1)} + \alpha_2 x_{k_3}^{(2)}\right] g_3.$$

Choosing $\alpha = e_1 = (1,0,0)^\top$ gives $0 = x_{k_2}^{(1)} g_2 = x_{k_3}^{(1)} g_3$. So, we obtain $g_2 = g_3 = 0$. Likewise, taking $\alpha = e_2 = (0,1,0)^\top$ yields $g_1 = 0$. Hence $g_1 = g_2 = g_3 = 0$, and consequently $\nabla\mathcal{A}\left(x^{(r)}\right)^m = 0$ for $r = 1,2,3$.

Since $\mathcal{A}$ is almost copositive-plus, by Lemma 2.1, there exists $x^* \geq 0$ such that $\mathcal{A}(x^*)^m = 0$ and $\nabla\mathcal{A}(x^*)^m \neq 0$.

Case 1. $k_0 \notin \{k_1, k_2, k_3\}$.

Define $z_r(t) = (1-t)x^{(r)} + tx^* > 0$ and set $f_r(t) = \mathcal{A}\left(z_r(t)\right)^m$, a polynomial in t of even degree $m \geq 4$. Then $f_r(t) \geq 0$ for all $t \in [0,1]$, and in particular $f_r(0) = f_r(1) = 0$.

Because 0 and 1 are multiple roots, we have $f_r'(0) = f_r'(1) = 0$. From $f_r'(t) = m\left(x^* - x^{(r)}\right)^\top \nabla\mathcal{A}\left(z_r(t)\right)^m = 0$ it follows that $\left(x^{(r)}\right)^\top g = (x^*)^\top g$ for $r = 1,2,3$, where $g = \nabla\mathcal{A}(x^*)^m$.

The vectors $x^{(1)}, x^{(2)}, x^{(3)}$ are linearly independent; therefore $g = 0$, a contradiction.

Case 2. $k_0 \in \{k_1, k_2, k_3\}$. Without loss of generality, assume $k_0 = k_1$.

Arguing similarly to Case 1, we obtain

$$\left(x^{(2)}\right)^\top g = (x^*)^\top g, \left(x^{(3)}\right)^\top g = (x^*)^\top g.$$

Since $(x^*)^{\top g} = m\mathcal{A}(x^*)^m = 0$, we have

$$\left(x^{(2)}\right)^\top g = \left(x^{(3)}\right)^\top g = 0 \tag{*}$$

Because x^* is 0 only at the component k_1 and positive elsewhere, $g_k = (\nabla\mathcal{A}(x^*)^m)_k = 0\ \forall k \neq k_1$; hence $g = \lambda e_{k_1}$ for some scalar λ. Then $\left(x^{(2)}\right)^\top g = \lambda x_{k_1}^{(2)}$. As $k_1 \neq k_2$ and $x_{k_1}^{(2)} > 0$, equation (*) forces $\lambda = 0$, i.e., $g = 0$ contradiction.

In conclusion, $\mathcal{A}$ can have at most two key indices.

3. Conclusion and Future Work

This paper extends the theory of almost copositive matrices to even-order real symmetric tensors. We first introduced the definitions of almost copositive, almost copositive-plus, and almost strictly copositive tensors, and introduced the concept of key indices as an important tool for characterizing the structure of almost copositive-plus tensors. By constructing a constrained homogeneous polynomial optimization problem and applying the KKT conditions, we proved that an almost copositive tensor always possesses a negative eigenvalue corresponding to a positive eigenvector, while also revealing the existence of key indices for almost copositive-plus tensors and establishing an upper bound on their number. These results not only uncover structural features of higher-order tensors with respect to almost copositive properties, but also provide new theoretical insights into the eigenvalue distribution of tensors and the feasibility of polynomial optimization problems.

The present work opens up new research directions in tensor optimization theory; however, several questions remain worthy of further investigation: the uniqueness of key indices and whether other properties of almost copositive matrices continue to hold in the tensor setting. Addressing these issues will contribute to a more complete establishment of the theory of almost copositive tensors.

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