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Matrix Riccati Equations in Optimal Control: A Differential Transform Method Approach

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Abstract

In this article, we investigate an application of the matrix Riccati equation (MRE) arising in the linear-quadratic regulator (LQR), a fundamental class of optimal control problems. The characterization of the control in the LQR setting depends critically on the integrability of the MRE. Unfortunately, there is no general way of solving the MRE. An analytic solution or closed-form solution to the MRE depends on how the coefficients of the equation are connected to each other. To address this limitation, we derive semi-analytic series solutions to the MRE by leveraging the power of the differential transform method (DTM). This approach yields an explicit series representation of both the Riccati solution and the corresponding optimal control. We further investigate the convergence properties of the proposed approximation by analyzing the logarithm of the squared approximation error (log-squared error), thereby providing quantitative insight into the accuracy and stability of the series solution.

Keywords

Matrix Riccati Equation; Linear Quadratic Regulator; Differential Transform Method

1. Introduction

The matrix Riccati equation, named after the mathematician Jacopo Francesco Riccati [1], can be written as

$$\frac{dY(t)}{dt} = Y(t)A(t)Y(t) + Y(t)B(t) + C(t)Y(t) + D(t)$$

where $Y(t) \in \mathbb{R}^{n\times m}$, $A(t) \in \mathbb{R}^{m\times n}$, $B(t) \in \mathbb{R}^{m\times m}$, $C(t) \in \mathbb{R}^{n\times n}$, $D(t) \in \mathbb{R}^{n\times m}$.

The matrix Riccati equation arises in many branches of applied mathematics [2], including Optimal control [3], stabilization theory, transport theory, physics, filtration of control systems, differential games [4, 5], random processes, diffusion problems, non-uniform transmission lines, and stochastic control.

The scalar Riccati equation, defined as

$$\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t)$$

where $a(t)$, $b(t)$ and $c(t)$ are real value functions, is a particular case of the matrix Riccati equation. It appears to be the simplest non-linear differential equation. Similar to the matrix Riccati equation, there is no general way of solving the scalar Riccati equation. We mostly rely on a change of variables to find an analytic solution. One of the most well-known change of variables consists of transforming the Riccati equation into a second order linear differential equation.

For instance, the change of variable $y = -\frac{\frac{du}{dt}}{a(t)u}$ leads to the equation:

$$\frac{d^2u(t)}{dt^2} - \left(b(t) + \frac{\frac{da(t)}{dt}}{a(t)}\right)\frac{du(t)}{dt} + a(t)c(t)u(t) = 0$$

That can be solved in a general way if the coefficients are constant. This transformation is particularly useful in quantum mechanics and can also serve as a blueprint for solving the matrix Riccati equation, as demonstrated in Section 2. Another approach for solving the scalar Riccati equation and certain partial differential equations is the differential transform Method (DTM). The concept of the DTM was first introduced by Zhou [6], and its main purpose was to solve linear and non-linear initial value problems in electrical circuit analysis. The DTM is a numerical and analytical method for solving a wide variety of differential equations and partial differential equations. This method provides the solution of a variety of differential equations in terms of convergent series with easily computable components. However, it is different from the Taylor series method, which requires symbolic computation of derivatives and involves more computations, especially for higher-order differential equations.

In this article, we use the scalar Riccati equation as a blueprint for how we should move forward to semi-analytically solve the matrix Riccati equation. We look at applying the DTM to the matrix Riccati equation, which requires extending the properties of the DTM to matrices. We consider one of the many applications of matrix Riccati equations, the time-varying linear quadratic regulator, which is a particular case of an optimal control problem.

As mentioned above, the matrix Riccati equation arises in optimal control theory [7], particularly in the linear quadratic regulator (LQR). The LQR, in which the dynamics are linear and the cost functions are quadratic, appears to be the simplest case of dynamic programming problems for continuous systems [8-10]. The LQR remains one of the most well-studied problems in optimal control, not just because of the simplicity of the model in terms of its computation, but because it is involved in so many applications. For instance, they are present in circuits [11], in robotics [12], in economic stabilization policy [13], in solving aviation problems such as the full tracking problem in aircraft system identification, and control [14].

Unlike most optimal control problems, which rely on numerical methods to estimate the value of the control, the LQR remarkably expresses the control analytically in terms of the solution of a matrix Riccati equation. Therefore, solving the MRE plays a crucial part in finding the control. Notice that several variants of the LQR have been considered in the past, along with different algorithms. For instance, the finite-horizon, invariant, continuous-time LQR was considered in several papers. In [15], the authors use a technique that consists of transforming the MRE into a Lyapunov differential equation whose solution can be derived using the tensor product. The infinite-horizon, invariant-time, discrete-time, or continuous-time has been studied by several authors [16-18]. Unlike the finite-horizon approach, the control is computed using the algebraic matrix Riccati equation. In this study, we consider the finite-horizon, time-variant, continuous-time LQR. The control can be calculated analytically in terms of the solution of a Riccati equation. We will leverage the power of the DTM to derive the solution of the Riccati equation, which leads to an explicit expression of the control.

The article is organized as follows: In section 2, we consider the LQR, which seems to yield the most important and influential result in optimal control to date. We give an example of an LQR that can be solved thoroughly using a change of variables. In Section 3, the differential transform method is considered. We extend some of the properties of the DTM to matrix Riccati equations and provide an example of LQR that is completely solved using the DTM. Additionally, we showcase graphs that demonstrate the potential implementation of the DTM and analyze the convergence of its error. Section 4 is dedicated to the conclusion.

2. Linear Quadratic Regulator

2.1 Derivation of Solution to LQR

The linear quadratic regulator (LQR) minimizes a quadratic cost functional

$$J\big(x(.), u(.)\big) = \frac{1}{2}\int_{t_0}^{t_f}(x(t)^T Q(t)x(t) + u(t)^T R(t)u(t))dt \tag{2.1}$$

subject to a linear ordinary differential equation constraint

$$\frac{dx}{dt} = A_1(t)x(t) + B_1(t)u(t) \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control, $Q(t) \in \mathbb{R}^{n\times n}$ is symmetric and semi-positive definite, $R(t) \in \mathbb{R}^{m\times m}$ symmetric and positive definite, $A_1(t) \in \mathbb{R}^{n\times n}$, $B_1(t) \in \mathbb{R}^{n\times m}$ and $x(t_0) = x_0$. The goal is to find the optimal control function $u(t)$ that minimizes $J\big(x(t), u(t)\big)$.

From the Pontryagin Maximum Principle [3],

$$u(t) = -R^{-1}(t)B_1^T(t)P(t)x(t) \tag{2.3}$$

where P (t) satisfies the Riccati equation:

$$\frac{dP}{dt} = P(t)B_1(t)R^{-1}(t)B_1^T(t)P(t) - A_1^T(t)P(t) - P(t)A_1(t) - Q(t) \tag{2.4}$$

with initial condition $P\big(t_f\big) = 0$.

2.2 LQR Example: Change of Variable

This example makes use of the theorem from the appendix to solve an LQR problem. We have the given matrices:

$$A_1 = R = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B_1 = \begin{pmatrix} e^{\frac{t}{2}} & -e^{\frac{t}{2}} \\ e^{\frac{t}{2}} & e^{\frac{t}{2}} \end{pmatrix},\ Q = \begin{pmatrix} \frac{e^{-x}}{2} & 0 \\ 0 & \frac{e^{-x}}{2} \end{pmatrix} \tag{2.5}$$

When plugging (2.5) into (2.4), we get the matrix Riccati equation,

$$\frac{dP}{dt} = P(2e^tI_2)P - 2P - \frac{1}{2}e^tI_2.$$

Using the change of variable (see appendix)

$$P = -(2e^tI_2)^{-1}V'V^{-1}$$

where $V \in \mathbb{R}^{2\times 2}$ and invertible,
we can turn (2.4) into a constant coefficient second-order matrix differential equation of the form

$$V'' + V' - V = 0$$

where $V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$
therefore

$$V_{11} = k_1e^{r_1t} + k_2e^{r_2t},\ V_{12} = k_3e^{r_1t} + k_4e^{r_2t},\ V_{21} = k_5e^{r_1t} + k_6e^{r_2t},\ V_{22} = k_7e^{r_1t} + k_8e^{r_2t}$$

where $r_1 = \frac{-1+\sqrt{5}}{2}$, $r_2 = \frac{-1-\sqrt{5}}{2}$. Notice that P(1) = 0.
$V^{-1}(1) \neq 0$, since V is invertible, then $V'(1)$ must be the zero matrix.

Since $V'(t) = \begin{pmatrix} k_1r_1e^{r_1t} + k_2r_2e^{r_2t} & k_3r_1e^{r_1t} + k_4r_2e^{r_2t} \\ k_5r_1e^{r_1t} + k_6r_2e^{r_2t} & k_7r_1e^{r_1t} + k_8r_2e^{r_2t} \end{pmatrix}$

Therefore, $k_1 = -k_2\frac{r_2}{r_1}e^{r_2-r_1},\ k_3 = -k_4\frac{r_2}{r_1}e^{r_2-r_1},\ k_5 = -k_6\frac{r_2}{r_1}e^{r_2-r_1}, k_7 = -k_8\frac{r_2}{r_1}e^{r_2-r_1}$.

The formula of $V(t)$ $V'(t)$ and $V^{-1}(t)$ follows:

$$V(t) = \left(-\frac{r_2}{r_1}e^{r_2-r_1}e^{r_1t} + e^{r_2t}\right)\begin{pmatrix} k_2 & k_4 \\ k_6 & k_8 \end{pmatrix}$$

$$V'(t) = \left(-r_2e^{(r_2-r_1)}e^{r_1t} + r_2e^{r_2t}\right)\begin{pmatrix} k_2 & k_4 \\ k_6 & k_8 \end{pmatrix}$$

$$V^{-1}(t) = \frac{1}{\left(-\frac{r_2}{r_1}e^{r_2-r_1}e^{r_1t}+e^{r_2t}\right)^2(k_2k_8-k_4k_6)}\begin{pmatrix} k_8 & -k_4 \\ -k_6 & k_2 \end{pmatrix}.$$

Therefore, a solution of the Riccati equation (2.4) is given by:

$$P(t) = -\left(\frac{e^{-t}}{2} I_2\right) \frac{\left(-r_2 e^{(r_2-r_1)} e^{r_1 t} + r_2 e^{r_2 t}\right)}{\left(-\frac{r_2}{r_1} e^{r_2 - r_1} e^{r_1 t} + e^{r_2 t}\right)^2 (k_2 k_8 - k_4 k_6)} \begin{pmatrix} k_2 & k_4 \\ k_6 & k_8 \end{pmatrix} \begin{pmatrix} k_8 & -k_4 \\ -k_6 & k_2 \end{pmatrix}.$$

After cancellation

$P(t) = -\left(\frac{e^{-t}}{2}\right) \frac{\left(-r_2 e^{(r_2-r_1)} e^{r_1 t} + r_2 e^{r_2 t}\right)}{\left(-\frac{r_2}{r_1} e^{r_2 - r_1} e^{r_1 t} + e^{r_2 t}\right)^2} I_2$ and the control $u(t) = \left(\frac{e^{-t}}{2}\right) \frac{\left(-r_2 e^{(r_2-r_1)} e^{r_1 t} + r_2 e^{r_2 t}\right)}{\left(-\frac{r_2}{r_1} e^{r_2 - r_1} e^{r_1 t} + e^{r_2 t}\right)^2} B_1^T(t) x(t)$

3. Differential Transform Method

The differential transform method (DTM) is a numerical and analytical method to solve a diverse range of differential equations, providing a solution in terms of a series. The DTM has been used to find explicit solutions to the scalar Riccati equation, as expressed in terms of a Taylor series expansion [6].

For reference, the kth order differential transform of a function $f(x)$ about a point x_0 is defined as

$$F(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k} f(x)\right]_{x=x_0} \tag{3.1}$$

where k is a nonnegative integer.

The inverse differential transform is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k$$

$$f(\mathrm{x}) = \sum_{\mathrm{k}=0}^{\infty} \frac{(x-x_0)^k}{k!}\left[\frac{d^k}{dx^k} f(x)\right]_{x=x_0} \tag{3.2}$$

Combining (3.1) and (3.2) results in a Taylor series for $f(x)$ about $x = x_0$.

We can also extend this definition of DTM to a matrix as follows:

If $f(x) = A(x)$ where $A(x) \in \mathbb{R}^{n\times m}$, then, the kth order differential transform of $A(x)$ about x_0 is defined as

$$F(k) = \left(\frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} a_{ij}(x)\right]_{x=x_0}\right)_{1\le i\le n, 1\le j\le m}$$

where $a_{ij}(x)$ is the (i, j) −entry of $A(x)$

3.1 Properties of the Differential Transform

Using the definition of the differential transform, the following properties can be derived for a real-valued function $f(x)$ and for matrices as well.

1. If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$
2. If $f(x) = \lambda g(x)$, then $F(k) = \lambda G(k)$, where λ is a constant
3. If $f(x) =- \frac{\partial g(x)}{\partial x}$ then $F(k) = (k + 1)\, G(k + 1)$
4. If $f(x) = \frac{\partial^m}{\partial x^m} g(x)$ then $F(k) = (k + 1)(k + 2) \ldots (k + m)\, G(k + m)$
5. If $f(x) = x^m$, then $F(k) = \delta(k-m) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$
6. If $f(x) = g(x)h(x)$, then $F(k) = \sum_{r=0}^{k} G(r)H(k-r)$
7. If $f(x) = f_1(x) f_2(x) \ldots f_m(x)$, then
$$F(k) = \sum_{k_{m-1=0}}^{k} \sum_{k_{m-2}=0}^{k_{m-1}} \ldots \sum_{k_1=0}^{k_2} F_1(K_1)F_2(k_2 - k_1) \ldots F_m(k - k_{m-1})$$
8. If $f(x) = e^{Bx}$ then $F(k) = \frac{B^k}{k!}$
9. If $f(x) = \int_0^x g(t)dt$, then $F(k) = \frac{G(k-1)}{k}$, $k \ge 1$, $F(0) = 0$

10. If $f(x) = \sin(\varpi x + \alpha)$, then $F(k) = \frac{\varpi^k}{k!}\sin\left(\frac{\pi k}{2} + \alpha\right)$
11. If $f(x) = \cos(\varpi x + \alpha)$, then $F(k) = \frac{\varpi^k}{k!}\cos\left(\frac{\pi k}{2} + \alpha\right)$

3.2 Example as applied to Optimal Control

We consider the matrix Riccati equation

$$\frac{dP}{dt} = PB_1R^{-1}B_1^TP - A_1^TP - PA_1 - Q$$

where B_1, R, A_1, Q are all functions of t, with initial condition

$$P(t_f) = 0$$

We choose an example that can be applied to optimal control, specifically the LQR. Thus, we define the coefficient matrices abiding by the aforementioned conditions that allow for the LQR problem to be solved using the Riccati equation. We choose:

$$R(t) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B_1(t) = \begin{pmatrix} \cos\,(t - t_{f)} & -\sin(t - t_f) \\ \sin(t - t_f) & \cos(t - t_f) \end{pmatrix}$$

$$A_1(t) = \begin{pmatrix} \frac{1}{2}(-e^{t-t_f} + 1) & \frac{1}{4}(t - t_f) \\ \frac{1}{4}(t - t_f) & \frac{1}{2}\left(e^{-(t-t_f)} - 1\right) \end{pmatrix}$$

In the case, we recognize that $Q(t) = -D(t) = -\frac{dP}{dt}$, with $D(t)$ referring to the original form of matrix Riccati equation. Thus

$$Q(t) = \begin{pmatrix} e^{t-t_f} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & e^{-(t-t_f)} \end{pmatrix}$$

So, $R(t)$ is symmetric and $Q(t)$ is positive semi-definite. B_1 is not necessarily positive definite, but rather is defined so that $B_1B_1^T = I_2$. We introduce the change of variable $G(t) = P(t + t_f)$ so it follows that $G(0) = P(t_f) = 0$. The Riccati equation becomes:

$$\frac{dG}{dt} = GG - A_2^TG - GA_2 - Q_2$$

Where $A_2 = \begin{pmatrix} \frac{1}{2}(-e^t + 1) & \frac{1}{4}t \\ \frac{1}{4}t & \frac{1}{2}(e^{-t} - 1) \end{pmatrix}$ and $Q_2(t) = \begin{pmatrix} e^t & -\frac{1}{2} \\ -\frac{1}{2} & e^{-t} \end{pmatrix}$

Let T be the differential transform of G of kth order such that $T(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k}G(t)\right]_{t=0}$

with $T(0) = \frac{1}{0!}G(0) = 0$.

Let F_1 and F_2 be the differential transforms of A_2 and Q_2 respectively. Applying the differential transform to both sides of the matrix Riccati equation yields the recurrence relation.

$$(k+1)T(k+1) = \sum_{k_1=0}^{k} T(k_1)T(k-k_1) - \sum_{k_1=0}^{k} F_1^T(k_1)T(k-k_1) - \sum_{k_1=0}^{k} T(k_1)F_1(k-k_1) - F_2(k)$$

$$(k+1)\begin{pmatrix} T_{11}(k+1)T_{12}(k+1) \\ T_{21}(k+1)T_{22}(k+1) \end{pmatrix} = \sum_{k_1=0}^{k} \begin{pmatrix} T_{11}(k_1)T_{12}(k_1) \\ T_{21}(k_1)T_{22}(k_1) \end{pmatrix} \begin{pmatrix} T_{11}(k-k_1)T_{12}(k-k_1) \\ T_{21}(k-k_1)T_{22}(k-k_1) \end{pmatrix}$$

$$-\sum_{k_1=0}^{k}\begin{pmatrix}\frac{1}{2}\left(-\frac{1}{k!}+\delta(k_1-0)\right) & \frac{1}{4}\delta(k_1-1)\\ \frac{1}{4}\delta(k_1-1) & \frac{1}{2}\left(-\frac{(-1)^{k_1}}{k!}-\delta(k_1-0)\right)\end{pmatrix}\begin{pmatrix}T_{11}(k-k_1) & T_{12}(k-k_1)\\ T_{21}(k-k_1) & T_{22}(k-k_1)\end{pmatrix}$$

$$-\sum_{k_1=0}^{k}\begin{pmatrix}T_{11}(k_1) & T_{12}(k_1)\\ T_{21}(k_1) & T_{22}(k_1)\end{pmatrix}\begin{pmatrix}\frac{1}{2}\left(-\frac{1}{(k-k_1)!}+\delta(k-k_1-0)\right) & \frac{1}{4}\delta(k-k_1-1)\\ \frac{1}{4}\delta(k-k_1-1) & \frac{1}{2}\left(-\frac{(-1)^{k-k_1}}{(k-k_1)!}-\delta(k-k_1-0)\right)\end{pmatrix}$$

$$-\begin{pmatrix}\frac{1}{k!} & -\frac{1}{2}\delta(k-0)\\ -\frac{1}{2}\delta(k-0) & \frac{(-1)^k}{k!}\end{pmatrix}$$

If $k=0$ then

$$T(1)=\sum_{k_1=0}^{0}T(k_1)T(0-k_1)-\sum_{k_1=0}^{0}F_1^T(k_1)T(0-k_1)-\sum_{k_1=0}^{0}T(k_1)F_1(0-k_1)-F_2(0)$$
$$=T(0)T(0)-F_1^T(0)T(0)-T(0)F_1(0)-F_2(0)$$

Since $T(0)=0$ then $T(1)=-F_2(0)=-\begin{pmatrix}\frac{1}{0!} & -\frac{1}{2}\delta(0-0)\\ -\frac{1}{2}\delta(0-0) & \frac{(-1)^0}{0!}\end{pmatrix}=\begin{pmatrix}-1 & \frac{1}{2}\\ \frac{1}{2} & -1\end{pmatrix}$

We can use the recurrence relation to find $T(2),T(3),T(4)$.
If $k=1$ then

$$2T(2)=\sum_{k_1=0}^{1}T(k_1)T(k-k_1)-\sum_{k_1=0}^{1}F_1^T(k_1)T(k-k_1)-\sum_{k_1=0}^{1}T(k_1)F_1(k-k_1)-F_2(1)$$
$$=\big(T(0)T(1-0)+T(1)T(0)\big)-\big(F_1^T(0)T(1-0)+F_1^T(1)T(0)\big)$$
$$-\big(T(0)F_1(1-0)+T(1)F_1(0)\big)-F_2(0)$$

Since $T(0)=F_1(0)=0$ then $2T(2)$=-$F_2(1)$

$$T(2)=-\frac{1}{2}\begin{pmatrix}\frac{1}{1!} & -\frac{1}{2}\delta(1-0)\\ -\frac{1}{2}\delta(1-0) & \frac{(-1)^1}{1!}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$$

We compute the same way $T(3)=\begin{pmatrix}-\frac{1}{3!} & 0\\ 0 & -\frac{1}{3!}\end{pmatrix}$ and $T(4)=\begin{pmatrix}-\frac{1}{4!} & 0\\ 0 & -\frac{1}{4!}\end{pmatrix}$

From here, we may write the differential inverse as:

$$G(t)=\sum_{k=0}^{\infty}T(k)t^k=T(0)+T(1)t+T(2)t^2+T(3)t^3+\cdots$$

$$=\begin{pmatrix}-t-\frac{1}{2}t^2-\frac{1}{3!}t^3-\frac{1}{4!}t^4-\cdots & \frac{1}{2}t\\ \frac{1}{2}t & -t+\frac{1}{2}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\cdots\end{pmatrix}$$

$$= \begin{pmatrix} -\sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!} & \frac{1}{2}t \\ \frac{1}{2}t & \sum_{k=0}^{\infty} \frac{(-t)^{k+1}}{(k+1)!} \end{pmatrix}$$

We identify the series $-t - \frac{1}{2}t^2 - \frac{1}{3!}t^3 - \frac{1}{4!}t^4 - \cdots$ and $-t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \cdots$ as the series of $-e^t + 1$ and $e^{-t} - 1$, respectively. Therefore, $G(t)$ may be given by

$$G(t) = \begin{pmatrix} -e^t + 1 & \frac{1}{2}t \\ \frac{1}{2}t & e^{-t} - 1 \end{pmatrix}$$

This can be verified by replacing $G(t)$ in the matrix Riccati equation:

$$\frac{dG}{dt} = GG - A_2^T G - GA_2 - Q_2$$

Moreover, $G(0) = 0$, yielding the value of $P(t)$

$$P(t) = G(t - t_f) = \begin{pmatrix} -e^{t-t_f} + 1 & \frac{1}{2}(t - t_f) \\ \frac{1}{2}(t - t_f) & e^{-(t-t_f)} - 1 \end{pmatrix}$$

Thus, it follows that the control may be written as $u(t) = -R^{-1}(t)B_1^T(t)P(t)x(t)$. Since $R^{-1}(t) = I_2$ therefore $u(t) = -B_1^T(t)P(t)x(t)$

Substituting in the respective matrices for $B_1^T(t)$ and $P(t)$ yields the expression for the control as

$$u(t) = -\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} -e^{t-t_f} + 1 & \frac{1}{2}(t - t_f) \\ \frac{1}{2}(t - t_f) & e^{-(t-t_f)} - 1 \end{pmatrix} x(t)$$

3.3 Numerical Application

The DTM was implemented in Python to approximate the solution to the matrix Riccati equation.

The example in section 3.2 was used to check the accuracy of the approximated $P(t)$, the full state feedback, with a chosen t_f. Letting $t_0 = 0$ and $t_f = 1$, the 4-term approximation was compared to the actual solution.

Let $P_{ij}(t)$ be the (i, j) −entry of $P(t)$. Due to $P_{12}(t) = P_{21}(t)$ being a polynomial of degree 1, its approximation Is equal to the actual for 4 terms. As the number of terms increases, the approximation of the actual solution improves. The 16-term approximation was also compared.

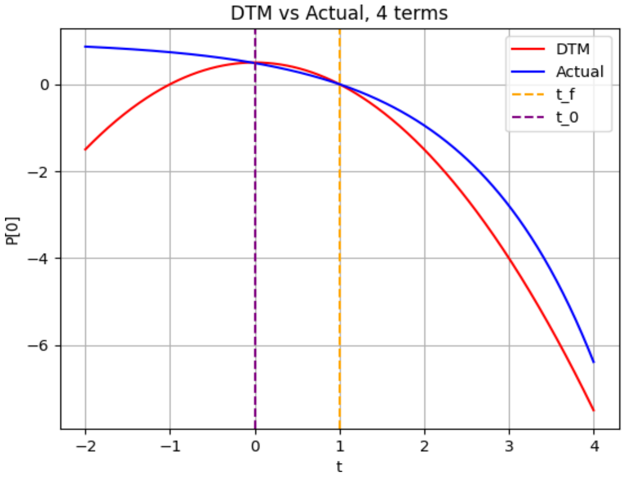

Figure 1. $P_{11}(t)$ Plot of 4-term approximation and actual solution.

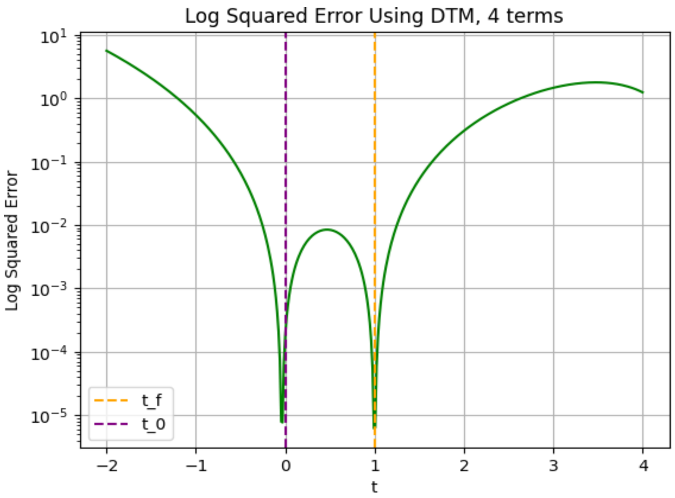

Figure 2. P11(t) Log-squared Error between 4-term approximation and actual solution.

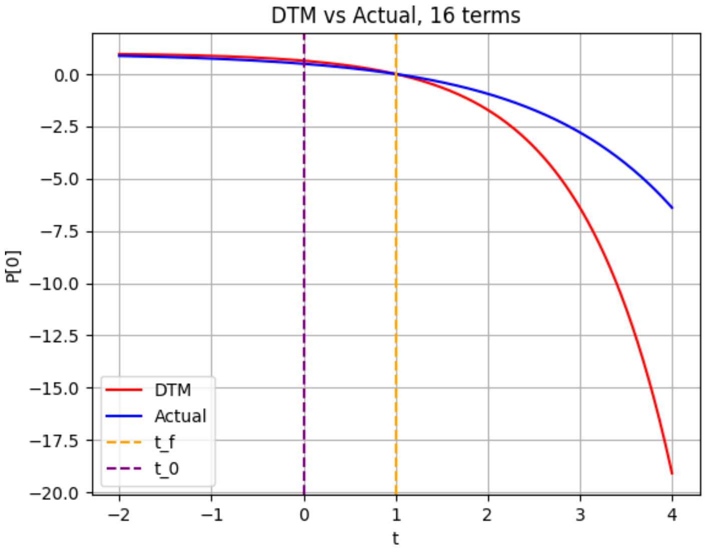

Figure 3. P_{11} Plot of 16-term approximation and actual solution.

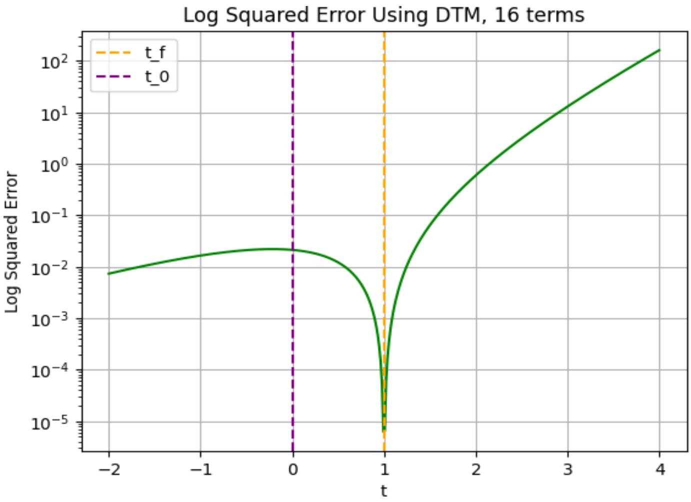

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Figure 4. P_{11} Log-squared Error between 16-term approximation and actual solution.

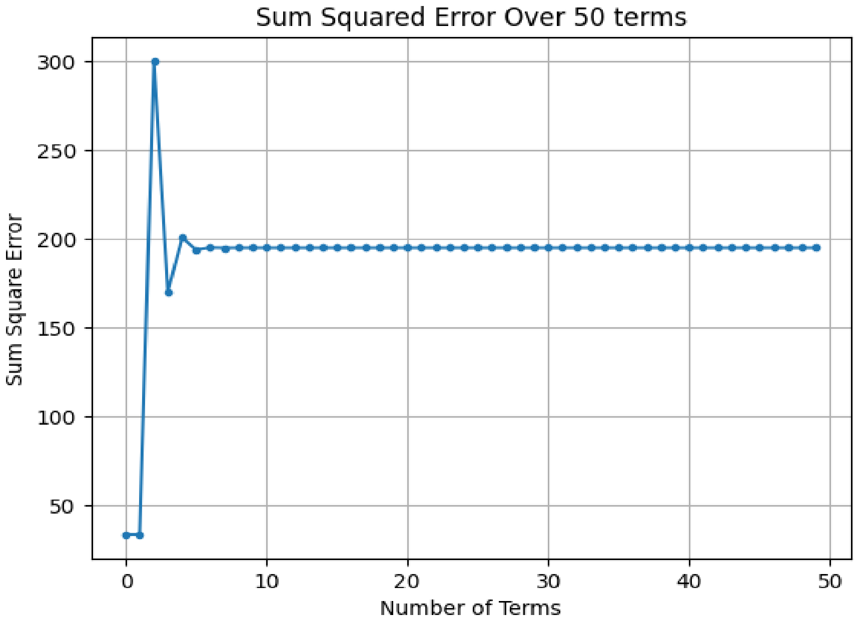

Figure 5. P11(t) Graph of sum squared error over 50 terms for t0 ≤ t ≤ tf.

We can estimate the norm overall error $R_k = G(t) - G_k(t)$ using the differential transform.
where $G_k(t)$ is the k-term Taylor series approximation of $G(t)$.
Then, using the Lagrange error bound for each entry,

$$\left|(R_k)_{ij}\right| \leq \max_{0\leq z\leq t}\left|\frac{d^{k+1}}{dt^{k+1}}G_{ij}(z)\right|\frac{|t|^{k+1}}{(k+1)!} = \max_{0\leq z\leq t}|T(k+1)||t|^{k+1}$$

Next is a bound of the matrix 1-norm of R_k

$$||R_k||_1 = \max_{1\leq j\leq n}\sum_{i=1}^{n}\left|(R_k)_{ij}\right| \leq \max_{1\leq j\leq n}\sum_{i=1}^{n}\max_{0\leq z\leq t}|T(k+1)||t|^{k+1}$$

4. Conclusion

The DTM has been used in solving, analytically and numerically a scalar Riccati equation and some partial differential equations. In this project, the DTM's properties are expanded for the purpose of solving the matrix Riccati equation. To do so, we utilized the link between the time-varying LQR and the MRE through the Pontryagin maximum principle. This resulted in an analytical expression of the control in an example of the LQR problem. Comparative analysis of a general graphic was done of the approximate value of the entries of the full state feedback of 4th and 16th order and the entries of exact full state feedback, with their respective errors. Conclusions can be derived regarding the potential, efficacy, and effectiveness of the DTM in dealing with the MRE and solving additional optimal control problems. An analysis of the convergence of its log-squared error was conducted in Section 3.3.

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Appendix: Solution using Familiar change of variable

Theorem: Consider the Riccati equation

$$\frac{dY}{dt} = YAY + CY + D$$

Using the change of variable $Y = -A^{-1}V'V^{-1}$, we can turn the Riccati equation into the second order linear matrix differential equation:

$$V'' - (A'A^{-1} + ACA^{-1})V' + ADV = 0$$

The proof is straightforward.