

Copositivity of a Class of 4th Order Cyclic Symmetric Tensors

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Abstract

In the context of the big data era, tensor structures capable of carrying high-order and high-dimensional information have increasingly become a research hotspot, and research on tensor theory and its applications across various fields has consequently flourished. Copositivity, as a key property in tensor theory, demonstrates wide application value in multiple important fields such as physics, optimization theory, and machine learning. Cyclic symmetric tensors are a class of tensors with high symmetry, and the study of their copositivity combines both theoretical distinctiveness and practical value. This paper systematically investigates the copositivity of a class of highly symmetric fourth-order three-dimensional cyclic symmetric tensors. Through in-depth analysis of this class of tensors under specific parameter structures, we establish a set of concise inequality relationships between their copositivity and their tensor components, thereby providing a systematic and practical analytical criterion for determining the copositivity of such tensors. This work not only expands the theoretical framework for tensor copositivity verification but also generalizes relevant results from existing research on tensor positive definiteness. Finally, based on the established tensor theoretical results, this paper further derives several (strict) inequalities concerning ternary quartic homogeneous polynomials.

Keywords

Cyclic symmetry; copositive tensors; homogeneous polynomials; ternary quartic forms

1. Introduction

Copositivity and positive definiteness are fundamental issues in tensor analysis and polynomial optimization, with significant applications in physics, engineering, and combinatorial optimization. In high-energy physics, one of the most direct applications of fourth-order tensors lies in the study of vacuum stability for Higgs scalar potential models [1-4]. Although verifying the copositivity of higher-order tensors is generally an NP-hard problem, for tensors with special symmetric structures—such as cyclic symmetry—analytically tractable necessary and sufficient conditions can be expected.

In 2013, Professor Liqun Qi first introduced the concept of copositive tensors, defining both copositivity and strict copositivity for symmetric tensors [5]. In 1988, Schmidt and Heß provided copositivity conditions for third-order two-dimensional symmetric tensors [6]. More recently, Qi, Song, and Zhang derived strict copositivity conditions for third-order two-dimensional tensors with the aid of Sturm's theorem [7]. Liu and Song further refined necessary and sufficient conditions for the copositivity of third-order two-dimensional symmetric tensors [4]. In recent years, progress has been made in the study of copositivity for fourth-order symmetric tensors. Qi first extended the notion of

positive definiteness to symmetric tensors and investigated it via H-eigenvalues and Z-eigenvalues [8]. To verify the copositivity and positive definiteness of higher-order tensors, the concept of cyclic symmetric groups was introduced, tracing back to the works of Stanley [9, 10], Yuan [11], Gao and Geroldinger [12], and Harris [13]. Subsequently, many researchers have focused on copositivity criteria for fourth-order two-dimensional symmetric tensors; for instance, Wang and Qi provided complete analytic conditions for the two-dimensional case [14]. Song established a criterion for the positive semi-definiteness of a class of fourth-order three-dimensional cyclic symmetric tensors [15]. Nevertheless, systematic necessary and sufficient conditions for the copositivity of tensors with the same structure remain lacking.

This paper aims to fill this gap by focusing on the copositivity of fourth-order cyclic symmetric tensors. We systematically establish analytically necessary and sufficient conditions for the copositivity and strict copositivity of a class of fourth-order three-dimensional cyclic symmetric tensors. Additionally, by applying the tensor results obtained above, several (strict) inequalities for ternary quartic homogeneous polynomials are derived. These results not only enrich the theoretical toolkit for verifying copositivity of higher-order tensors but also provide practical support for the analysis of related optimization problems and physical models.

2. Preliminaries

Throughout this paper, we use x^T to denote the transpose of a vector x . Tensors are denoted by uppercase calligraphic letters such as \mathcal{A} , \mathcal{B} , \mathcal{C} , ..., We consider only real tensors. The set of all m th-order n -dimensional real tensors is denoted by $T_{m,n}$, where m and n are positive integers with $m, n \geq 2$. A tensor $\mathcal{A} \in T_{m,n}$ is called **symmetric** if its entries $a_{i_1 \dots i_m}$ remain invariant under any permutation of their indices. The set of all m th-order n -dimensional real symmetric tensors is denoted by $S_{m,n}$.

For any $\mathcal{A} = (t_{i_1 \dots i_m}) \in T_{m,n}$ and $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ we define:

$$\mathcal{A}x^m = \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} \cdots x_{i_m}$$

This paper focuses on fourth-order symmetric tensors.

Definition 2.1. Let $\mathcal{A} = (t_{i_1 \dots i_m}) \in S_{m,n}$. The tensor \mathcal{A} is said to be

(i) **copositive** [5] if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}_+^n$.

(ii) **strictly Copositive** [5] if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}_+^n \setminus \{0\}$.

The concept of cyclic symmetry can be traced back to Stanley's research on cyclic symmetric groups, and has since been widely applied in tensor analysis [9-10].

Definition 2.2. A fourth-order three-dimensional symmetric tensor $\mathcal{A} = (a_{ijkl})$ is called

(i) **cyclic Symmetric** if

$$\begin{aligned} a_{1111} = a_{2222} = a_{3333}, a_{1112} = a_{2223} = a_{1333}, a_{1113} = a_{1222} = a_{2333} \\ a_{1122} = a_{1133} = a_{2233}, a_{1123} = a_{1223} = a_{1233} \end{aligned}$$

(ii) **totally Symmetric** if

$$\begin{aligned} a_{1111} = a_{2222} = a_{3333}, a_{1112} = a_{2223} = a_{1333} = a_{1113} = a_{1222} = a_{2333} \\ a_{1122} = a_{1133} = a_{2233}, a_{1123} = a_{1223} = a_{1233} \end{aligned}$$

Clearly, a totally symmetric tensor is a special case of a cyclic symmetric tensor.

3. Main Results

For a fourth-order three-dimensional cyclic symmetric tensor $\mathcal{A} = (a_{ijkl})$, its corresponding homogeneous polynomial has the standard form:

$$\begin{aligned} \mathcal{A}x^4 = a_{1111}(x_1^4 + x_2^4 + x_3^4) + 6a_{1122}(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 4a_{1112}(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) \\ + 4a_{1222}(x_1x_2^3 + x_1^3x_3 + x_2x_3^3) + 12a_{1123}x_1x_2x_3(x_1 + x_2 + x_3) \end{aligned}$$

Theorem 3.1. Let $\mathcal{A} = (a_{ijkl})$ be a fourth-order three-dimensional cyclic symmetric tensor. Suppose

$$a_{1111} = -a_{1123} = 1 \quad \text{and} \quad a_{1112}a_{1222} = -1.$$

Then, \mathcal{A} is copositive if and only if for all $i, j = 1, 2, 3$ with $i \neq j$, we have

$$a_{iijj} \geq \frac{11}{6}.$$

Furthermore, \mathcal{A} is strictly copositive if and only if for all $i, j = 1, 2, 3$ with $i \neq j$, we have

$$a_{iijj} > \frac{11}{6}.$$

Proof: Due to the cyclic symmetry of \mathcal{A} , we can assume without loss of generality that $a_{1222} = a_{2333} = a_{1113} = 1$ and $a_{1112} = a_{1333} = a_{2223} = -1$.

Necessity: From the copositivity of \mathcal{A} , taking $x = (1, 1, 1)^T$, we have

$$\begin{aligned} \mathcal{A}x^4 &= x_1^4 + x_2^4 + x_3^4 + 6a_{1122}(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 4(x_1x_2^3 + x_1^3x_3 + x_2x_3^3) \\ &\quad - 4(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) - 12(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) \\ &= (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) \\ &\quad + (6a_{1122} - 1)(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2 \\ &= 1^4 + 8 + 3 \times (6a_{1122} - 1) - 24 \geq 0 \end{aligned}$$

Thus, we obtain

$$a_{1122} \geq \frac{11}{6}.$$

By cyclic symmetry, we know $a_{1122} = a_{1133} = a_{2233}$. That is, for all $i, j = 1, 2, 3$ with $i \neq j$, we have $a_{iijj} \geq \frac{11}{6}$.

Sufficiency: Since $a_{iijj} \geq \frac{11}{6}$ and \mathcal{A} is cyclic symmetric, for any $x = (x_1, x_2, x_3)^T \in \mathbb{R}_+^3$, we have

$$\begin{aligned} \mathcal{A}x^4 &= x_1^4 + x_2^4 + x_3^4 + 4(x_1^3x_2 + x_1x_3^3 + x_3x_2^3) - 4(x_1^3x_3 + x_1x_2^3 + x_2x_3^3) \\ &\quad + 6a_{1122}(x_1^2x_2^2 + x_1x_3^2 + x_2x_3^2) - 12x_1x_2x_3(x_1 + x_2 + x_3) \\ &\geq x_1^4 + x_2^4 + x_3^4 + 4(x_1^3x_2 + x_1x_3^3 + x_3x_2^3) - 4(x_1^3x_3 + x_1x_2^3 + x_2x_3^3) \\ &\quad + 11(x_1^2x_2^2 + x_1x_3^2 + x_2x_3^2) - 12x_1x_2x_3(x_1 + x_2 + x_3) \\ &= \mathcal{A}'x^4 \end{aligned}$$

We can rewrite $\mathcal{A}'x^4$ as

$$\mathcal{A}'x^4 = (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) + 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 24x_1x_2x_3^2$$

Solve the constrained optimization problem:

$$\begin{aligned} \min \quad &\mathcal{A}'x^4 \\ \text{s. t.} \quad &x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

The minimum value of 0 is attained at the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, therefore, $\mathcal{A}x^4 \geq \mathcal{A}'x^4 \geq 0$. That is, \mathcal{A} is copositive.

If strict copositivity is required, then replace the corresponding " ≥ 0 " with " > 0 " in the above derivation.

Corollary 3.1. Let $\mathcal{A} = (a_{ijkl})$ be a fourth-order three-dimensional cyclic symmetric tensor. Suppose

$$a_{1111} = -a_{1123} = 1 \quad \text{and} \quad a_{1112} = a_{1222} = 1.$$

Then, \mathcal{A} is copositive if and only if for all $i, j = 1, 2, 3$ with $i \neq j$, we have

$$a_{iijj} \geq \frac{1}{2}.$$

Furthermore, \mathcal{A} is strictly copositive if and only if for all $i, j = 1, 2, 3$ with $i \neq j$, we have

$$a_{iijj} > \frac{1}{2}.$$

Corollary 3.2. Let $\mathcal{A} = (a_{ijkl})$ be a fourth-order three-dimensional cyclic symmetric tensor. Suppose

$$a_{1111} = -a_{1123} = 1 \quad \text{and} \quad a_{1112} = a_{1222} = -1.$$

Then, \mathcal{A} is copositive if and only if for all $i, j = 1, 2, 3$ with $i \neq j$, we have

$$a_{iijj} \geq \frac{19}{6}.$$

Furthermore, \mathcal{A} is strictly copositive if and only if for all $i, j = 1, 2, 3$ with $i \neq j$, we have

$$a_{iijj} > \frac{19}{6}.$$

Theorem 3.2. Let $\mathcal{A} = (a_{ijkl})$ be a fourth-order three-dimensional cyclic symmetric tensor. Suppose

$$a_{1111} = a_{1122} = 1 \quad \text{and} \quad a_{1112}a_{1222} = -1.$$

Then, \mathcal{A} is copositive if and only if for all $i, j, k = 1, 2, 3$ with $i \neq j, i \neq k, j \neq k$, we have

$$a_{iijk} \geq -\frac{7}{12}.$$

Furthermore, \mathcal{A} is strictly copositive if and only if for all $i, j, k = 1, 2, 3$ with $i \neq j, i \neq k, j \neq k$, we have

$$a_{iijk} > -\frac{7}{12}.$$

Proof: Due to the cyclic symmetry of \mathcal{A} , we can assume without loss of generality that $a_{1222} = a_{2333} = a_{1113} = 1$ and $a_{1112} = a_{1333} = a_{2223} = -1$.

Necessity: From the copositivity of \mathcal{A} , taking $x = (1, 1, 1)^T$, we have

$$\begin{aligned} \mathcal{A}x^4 &= x_1^4 + x_2^4 + x_3^4 + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 4(x_1x_2^3 + x_1^3x_3 + x_2x_3^3) \\ &\quad - 4(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) + 12a_{1123}(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) \\ &= (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) + 12(a_{1123} - 1)x_1x_2x_3(x_1 + x_2 + x_3) \\ &= 3^4 - 8 \times 3 + 36(t_{1123} - 1) \geq 0. \end{aligned}$$

Thus, we obtain

$$a_{1123} \geq -\frac{7}{12}.$$

By cyclic symmetry, we know $a_{1123} = a_{1223} = a_{1233}$, that is, for all $i, j, k = 1, 2, 3$ with $i \neq j, i \neq k, j \neq k$, we have $a_{iijk} \geq -\frac{7}{12}$.

Sufficiency: When $a_{1123} = a_{1223} = a_{1233} \geq -\frac{7}{12}$, \mathcal{A} can be rewritten as

$$\begin{aligned} \mathcal{A}x^4 &= (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) \\ &\quad + 12(a_{1123} - 1)x_1x_2x_3(x_1 + x_2 + x_3) \end{aligned}$$

Clearly, $x_1x_2x_3(x_1 + x_2 + x_3) \geq 0$.

At this point, $\mathcal{A}x^4$ is always greater than or equal to the value of $\mathcal{A}'x^4$ corresponding to $a_{1123} = -\frac{7}{12}$, i.e.,

$$\begin{aligned} \mathcal{A}x^4 &= (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) + 12(t_{1123} - 1)x_1x_2x_3(x_1 + x_2 + x_3) \\ &\geq (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) - 19x_1x_2x_3(x_1 + x_2 + x_3) \\ &= \mathcal{A}'x^4 \end{aligned}$$

Solve the constrained optimization problem:

$$\begin{aligned} \min \quad &\mathcal{A}'x^4 \\ \text{s. t.} \quad &x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

The minimum value of 0 is attained at the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, therefore $\mathcal{A}x^4 \geq \mathcal{A}'x^4 \geq 0$. That is, \mathcal{A} is copositive.

If strict copositivity is required, then replace the corresponding " ≥ 0 " with " > 0 " in the above derivation.

Corollary 3.3. Let $\mathcal{A} = (a_{ijkl})$ be a fourth-order three-dimensional cyclic symmetric tensor. Suppose

$$a_{1111} = a_{1122} = 1 \quad \text{and} \quad a_{1112} = a_{1222} = 1.$$

Then, \mathcal{A} is copositive if and only if for all $i, j, k = 1, 2, 3$ with $i \neq j, i \neq k, j \neq k$, we have

$$a_{iijk} \geq -\frac{5}{4}.$$

Furthermore, \mathcal{A} is strictly copositive if and only if for all $i, j, k = 1, 2, 3$ with $i \neq j, i \neq k, j \neq k$, we have

$$a_{iijk} > -\frac{5}{4}.$$

Corollary 3.4. Let $\mathcal{A} = (a_{ijkl})$ be a fourth-order three-dimensional cyclic symmetric tensor. Suppose

$$a_{1111} = -a_{1122} = 1 \quad \text{and} \quad a_{1112} = a_{1222} = 1.$$

Then \mathcal{A} is copositive if and only if for all $i, j, k = 1, 2, 3$ with $i \neq j, i \neq k, j \neq k$, we have

$$a_{iijk} \geq -\frac{1}{4}.$$

Furthermore, \mathcal{A} is strictly copositive if and only if for all $i, j, k = 1, 2, 3$ with $i \neq j, i \neq k, j \neq k$, we have

$$a_{iijk} > -\frac{1}{4}.$$

By applying the (strict) copositivity results for tensors established above, a series of (strict) inequalities for ternary quartic homogeneous polynomials can be derived.

Corollary 3.5. If $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$, then

$$\begin{aligned} (x_1 + x_2 - x_3)^4 &\geq 24x_1x_2x_3^2 - 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) - 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2); \\ (x_1 + x_2 - x_3)^4 &\geq 24x_1x_2x_3^2 + 8(x_1^3x_2 + x_1x_2^3) - 13(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2); \\ (x_1 + x_2 + x_3)^4 &\geq 3(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 24x_1x_2x_3(x_1 + x_2 + x_3); \\ (x_1 + x_2 + x_3)^4 &\geq 12(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 15x_1x_2x_3(x_1 + x_2 + x_3); \\ (x_1 + x_2 + x_3)^4 &\geq 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) + 19x_1x_2x_3(x_1 + x_2 + x_3); \\ (x_1 + x_2 + x_3)^4 &\geq 27x_1x_2x_3(x_1 + x_2 + x_3). \end{aligned}$$

The equality in each of these inequalities holds if and only if $x_1 = x_2 = x_3$. Furthermore, if we simultaneously exchange $x_1^3x_2$ with $x_1x_2^3$, $x_1^3x_3$ with $x_1x_3^3$, and $x_2^3x_3$ with $x_2x_3^3$, these inequalities still hold.

Corollary 3.6. If $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$, and $(x_1, x_2, x_3) \neq (0, 0, 0)$, then

$$\begin{aligned} (x_1 + x_2 - x_3)^4 &> 24x_1x_2x_3^2 - 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) - 5(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2); \\ (x_1 + x_2 - x_3)^4 &> 24x_1x_2x_3^2 + 8(x_1^3x_2 + x_1x_2^3) - 13(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2); \\ (x_1 + x_2 + x_3)^4 &> 3(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 24x_1x_2x_3(x_1 + x_2 + x_3); \\ (x_1 + x_2 + x_3)^4 &> 12(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 15x_1x_2x_3(x_1 + x_2 + x_3); \\ (x_1 + x_2 + x_3)^4 &> 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) + 19x_1x_2x_3(x_1 + x_2 + x_3); \\ (x_1 + x_2 + x_3)^4 &> 27x_1x_2x_3(x_1 + x_2 + x_3). \end{aligned}$$

Furthermore, if we simultaneously exchange $x_1^3x_2$ with $x_1x_2^3$, $x_1^3x_3$ with $x_1x_3^3$, and $x_2^3x_3$ with $x_2x_3^3$, these inequalities still hold.

4. Conclusions

This paper systematically studies the copositivity of fourth-order cyclic symmetric tensors, establishes analytic discriminative conditions for the three-dimensional case, and advances the theoretical development of such structured tensors.

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