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The Geometric Iteration Law of Dividing Any Angle into Three Equal Parts and Its Proof

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Abstract

The classical compass-and-straightedge method for trisecting an arbitrary angle is unsolvable when the corresponding arc is a segment of a circle, as it involves transcendental numbers. This study builds upon classical arguments while acknowledging the limitations of compass-and-straightedge constructions for direct trisecting. By utilizing the chord-arc relationship, we developed a construction method to approximate the one-third chord length of an arc, then iteratively generated segments equal to the chord length. Through extensive exploration, we successfully created two sets of construction diagrams that approximate the one-third chord length, enabling iterative generation of target segments. Using one set of these segments as special cases, we demonstrated that even under the constraint of trisecting the given arc, target segments can be iteratively produced. Since both proofs rely on numerical equality calculations—disregarding rigorous geometric principles—we employed geometric constructions to validate these methods.

Keywords

The corresponding arc to a given angle; the corresponding chord to a given angle; the one-third chord of the arc; the target line segment; constructing a line segment approximating the target line segment

1. Introduction

Let the special line segment equal one-third of the sum of the arc height and chord length be the upper base of an arc-constructed isosceles trapezoid, and use the upper base of this trapezoid

The sum of the two legs 'lengths constitutes one-third of the base of the next isosceles trapezoid. This method ensures that the third isosceles trapezoid's base and two legs can achieve integer values, thereby proving the theorem [1]. However, the title might be misinterpreted as "A Compass and Straightedge Can Construct a Point of Trisecting an Arc," and concerns about adaptability led to a revised paper titled "A Compass and Straightedge Can Construct a Line Segment Equal to One-Third of the Chord Length of the Arc Corresponding to a Given Angle" [2]. Although both sets of trapezoids achieve integer values for base and legs, they rely on computational data. Geometric proofs for trapezoids with unequal bases and legs evolving into isosceles trapezoids cannot be constructed due to the lack of established theories or theorems. Why is this? It turns out that certain geometric construction principles remain undiscovered. For instance, isosceles trapezoids on arcs corresponding to arbitrary angles follow specific patterns, which can serve as lemmas for geometric proofs of this construction.

Lemma 1: A trapezoid on any angle arc is an isosceles trapezoid; the sum of the three equal sides of a trapezoid is greater than that of the three unequal sides, and the segment with the exact equal sides of the trapezoid is the target segment, i.e., the one-third chord of the arc.

Lemma 2: For a trapezoid with unequal sides on any angle segment, one-third of the sum of its three sides can be constructed as a line segment, which is a fraction of the target line segment.

Lemma 3: For any angle arc, the constructed segment equaling one-third of the sum of the three sides of the trapezoid's upper base is an iteration segment. This iteration segment is within a fraction of a million to a fraction of a billion of the target segment.

Lemma 4: The segment which is one third of the sum of the three sides of the trapezoid with the upper base and is iterated on any arc is exactly equal to the target segment, the target segment is one third of the chord of the arc.

For instance, in a right triangle with equal legs, the sum of the three sides equals 3R. If the base is 0.9R, the legs measure 1.0488...R, and the sum of the two legs plus the base equals 2.9976...R, yielding a third of 0.9992. With the base set at 1.1R, the legs measure 0.9486...R, and the sum of the two legs plus the base equals 2.9973...R, resulting in a third of 0.9991. This method ensures that the maximum difference between the constructed segments of obtuse, right, and acute angles is less than a fraction of a thousandth of the target segment, hence termed a rule or lemma.

The lemma establishes that constructing a line segment with the trapezium's base equal to half the chord length serves as the foundation. When applying this method to a 180-degree arc, setting the base to 0.99R yields a segment precisely equal to R in a single iteration. Similarly, using 0.347 of the chord length as the base for a 60-degree angle produces a segment equal to 0.99999998... of the chord length, achieving exact equality with the 20-degree chord length in a single iteration.

According to the above geometric construction, it essentially defines a linear transformation. Based on the similarity in elementary geometry, this transformation has a unique fixed point, so it must coincide with it after a finite number of steps.

The discovery of the transformation law of the relationship between the top base and the side of the trapezoid on an arbitrary angle arc is made in this paper. The research process follows the Euclidean compass and straightedge postulates, and all the construction steps can be regarded as the finite composite of the basic constructions in the first to sixth volumes of Elements (such as drawing equal line segments, drawing perpendicular lines, taking the mean of proportions, etc.).

The construction of this paper is similar to the previous two papers, which are based on drawing the construction line segment equal to one-third of the sum of the three sides of an isosceles trapezoid on the given angle arc.

In iterative construction, the original method used the arc height plus one-third of the chord of a given angle as a special segment (referred to as "constructive segments" in this paper). Our approach involves drawing a perpendicular bisector of the chord's midpoint intersecting the arc, forming a triangle with one-third of the sum of its three sides. This method (where the sum of an isosceles triangle's three sides equals the sum of a trapezoid's three sides with a base equal to half the chord length) yields a segment three times longer than the target. However, it's one-third of the base's length that still falls short of the target. Through two iterations, the segment achieves integer-length equivalence with the target, though the proof relies on computation rather than rigorous geometry, making it a near-approximation. To achieve exact equality, we now define the constructive segment as one-third of the sum of an isosceles triangle's three sides formed by a perpendicular bisector of the chord's midpoint intersecting the arc. Although both segments are only a few thousandths shorter than the target, the former's total length increases from three times the target's length at 180 degrees to the sum of the base and height at 90 degrees, while our method's segment is merely a few thousandths shorter. Recent research on geometric proofs of trapezoidal iterations over arbitrary angles has revealed underlying patterns, though these have not been fully theorized. Through deeper analysis, we now formulate these principles as lemmas. Examples include large obtuse angles, small obtuse angles, right angles, and acute angles, with geometric proofs provided.

2. Theoretical Proof

2.1 The first case

Known number: $\angle AOB$(Obtuse angle) $\overrightarrow{OA} = \overrightarrow{OB}$

$\overrightarrow{AH} = \overrightarrow{BH} = \overrightarrow{AB}/2$

Request: $\angle AOC = \angle COD = \angle DOB = \angle AOB/3$

Drawing (see Figure 1)

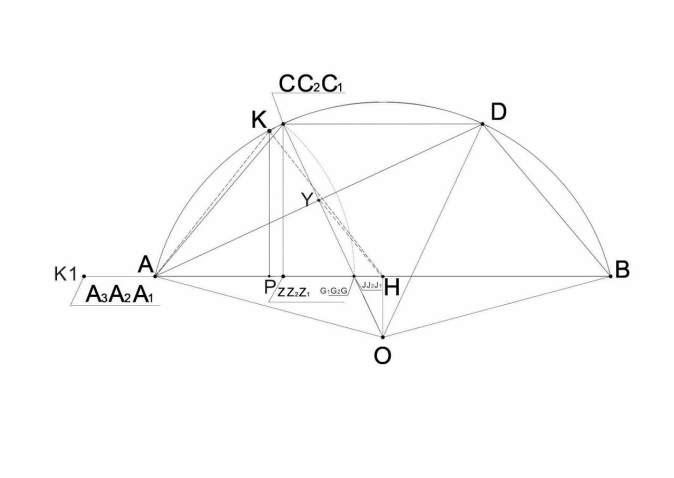

Figure 1. Proof of the Big Acute Angle by Third Division.

1. Do$\angle AOB$, $O\overrightarrow{OA}\widehat{AB}\overrightarrow{AB}$ draw a circle with the point as the center and the radius as the length, then connect the points.
2. Make a deal. $\overrightarrow{OH} \perp \overrightarrow{AB}\overrightarrow{AB}$ at the point H,
3. Draw $\overrightarrow{AH}$ the perpendicular bisectors of $\overrightarrow{AH}\widehat{AB}$ a line segment intersecting point P and point K, forming an isosceles triangle KAH.
4. Draw an P$\overrightarrow{AK}\overrightarrow{HA}K_1$arc with a point as the center and a radius, then extend the line of intersection to point
5. Three equal parts, =/$2\overrightarrow{K_1H}\overrightarrow{K_1Z_1} = \frac{2\overrightarrow{K_1H}}{32\overrightarrow{Z_1H}}, \overrightarrow{Z_1H}\overrightarrow{K_1H}/3\overrightarrow{K_1Z_1}$
6. The line $Z_1\overrightarrow{C_1Z_1} \perp \overrightarrow{AH}\widehat{AB}C_1$ passing through the point intersects at another point.
7. Draw an $Z_1\overrightarrow{A\ C_1HA}\overrightarrow{A_1}\overrightarrow{A_1H}\overrightarrow{Z_2H} = \overrightarrow{A_1H}/3$arc from a point as the center and radius, intersect the extension line at that point, then divide it into three equal parts.

$$\overrightarrow{A_1Z_2} = \frac{2\overrightarrow{A_1H}}{3} = 2\overrightarrow{Z_2H}$$

8. The line $Z_2\overrightarrow{C_2Z_2} \perp \overrightarrow{AH}\widehat{AB}C_2$ passing through the point intersects at another point.
9. Draw $Z_2\overrightarrow{A\ C_2HA}\overrightarrow{A_2}\overrightarrow{A_2H}\overrightarrow{ZH} = \overrightarrow{A_2H}/3 = \overrightarrow{A_2Z}/2$an arc from the center to the point where the extension of the radius intersects, and divide it into three equal parts.
10. Draw an arc from $\overrightarrow{A_2Z}\widehat{AB}$ at point C, iao$\overrightarrow{AH}$ point A with a radius equal to the distance from A to point J, intersecting point J.
11. The connecting $\overrightarrow{CO}\overrightarrow{AH}\angle AOC = \angle AOB/3$ line coincides with the intersection point J.

Prove: $\angle AOC = \angle AOB/3$

Create a guideline:

Draw an arc from point A with radius equal to 2, intersecting at point.

$\overrightarrow{K_1Z_1}\overrightarrow{AH}G_1\ \overrightarrow{AG_1}\overrightarrow{K_1Z_1} = \overrightarrow{Z_1H}$

Draw an arc from $\overrightarrow{A\ C_1}\overrightarrow{AH}J_1\ \overrightarrow{A\ C_1}\overrightarrow{A\ J_1}$ point A with radius equal to the distance from point A to the intersection point.

Draw an arc from point A with radius equal to 2, intersecting at point.

$\overrightarrow{A_1Z_2}\overrightarrow{AH}G_2\overrightarrow{AG_2}\overrightarrow{A_1Z_Z} = \overrightarrow{Z_2H}$

Draw an arc from $\overrightarrow{A\ C_2}\overrightarrow{AH}J_2\overrightarrow{A\ C_2}\overrightarrow{A\ J_2}$ point A with radius equal to the distance from point A to the intersection point.

The $\overrightarrow{CJ}\overrightarrow{CJ}$ at the point Y, iao$\widehat{AB}$ perpendicular bisectors of the triangle intersect at point D.

The connecting $\overrightarrow{OD}\overrightarrow{HB}\overrightarrow{CD}\ \overrightarrow{DB}$ lines intersect at point N.

Prove:

$\angle AOC = \angle COD = \angle DOB = \angle AOB/3$

$\because \overrightarrow{AP} + \overrightarrow{PH}\frac{\overrightarrow{AH}}{2}, \overrightarrow{KA} = \overrightarrow{KH},\ \overrightarrow{K_1P} + \overrightarrow{KA}$=$(\overrightarrow{KA} + \overrightarrow{AH} + \overrightarrow{KA})$/2,

$\therefore$ $(\overrightarrow{KA} + \overrightarrow{AH} + \overrightarrow{KA}) + \overrightarrow{AP}$,

$\because \overrightarrow{K_1P}\overrightarrow{AK}$
$\therefore \overrightarrow{AK}+\overrightarrow{PH}=\overrightarrow{(K_1P}+\overrightarrow{PH)/2}$,
$\because \overrightarrow{K_1Z_1}=2\ \overrightarrow{K_1H}\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}\overrightarrow{K_1H}\ \overrightarrow{K_1Z_1}=\overrightarrow{AG_1}$,
$\therefore \overrightarrow{AG_1}=\dfrac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}$, $\quad \overrightarrow{AG_1}$
For the target line segment,
Lemma 2: The Sum of the Three Sides of a Trapezoid with Unequal Sides on Any Angle Arc is one-third of the Total Sides
To construct a line segment, construct a line segment that is a few thousandths of the target line segment.
$\because \overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}$
In an isosceles triangle KAH, the three sides of the trapezoid formed by the top base of the arc trapezoid constructed on the base of the isosceles triangle are equal to
$\therefore \overrightarrow{AG_1}=\frac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}$ the requested constructed line segment
$\because \overrightarrow{AG=2(K_1Z_1+_1Z_1H)}/3,$
$\therefore$ One iteration of the sum of the three sides of a trapezoid divided by three One iteration of one − third the sum of the three sides of a trapezoid, one iteration of one $=2\overrightarrow{(A_1Z_1}\overrightarrow{Z_1H}\overrightarrow{AG_2}$,
$\because \overrightarrow{AG_2}\overrightarrow{AG_1}$
It constructs a segment equal to one-third of the sum of the three sides of the trapezoid, with the upper base as its base, which is a one-time iteration segment.
$\therefore \overrightarrow{AG_2}\overrightarrow{AG_1}$
As stated in Lemma 3, the construction line segment on any angle arc is one-third of the sum of the three sides of the upper trapezoid, which constitutes the iteration line segment. This iteration line segment is between a few millionths and a few hundredths of the target line segment. As stated in Lemma 3, the construction line segment on any angle arc is one-third of the sum of the three sides of the upper trapezoid, which constitutes the iteration line segment. This iteration line segment is between a few millionths and a few hundredths of the target line segment.
=2/3,=/3=/2,=, $\because \overrightarrow{A_1Z_2}\overrightarrow{A_1H}\ \overrightarrow{Z_2H}\overrightarrow{A_1H}\overrightarrow{A_1Z_2}\ \overrightarrow{A_1Z_2}\overrightarrow{AG_2}$
/3=+)/3, $\therefore \overrightarrow{AG_2}=2\overrightarrow{A_1H}2\overrightarrow{(AC_1}\overrightarrow{Z_1H}$
Greater than or equal to /3=+) / 3, +) /3,$\because \overrightarrow{AC_2}\overrightarrow{AC_1}$, $\overrightarrow{AG_2}2\overrightarrow{A_1H}2\overrightarrow{(AC_2}\overrightarrow{Z_2H}\overrightarrow{AG_1}=2\overrightarrow{(AC_1}\overrightarrow{Z_1H}$
+) / 3 is greater than +) / 3,$\therefore$ $2\overrightarrow{(AC_2}\overrightarrow{Z_2H}2\overrightarrow{(AC_1}\overrightarrow{Z_1H}$
$\because \overrightarrow{AG}2\overrightarrow{(AC_2}\overrightarrow{Z_2H}\ \overrightarrow{AG}\overrightarrow{AG}\overrightarrow{AC}$=+) / 3 is the second iteration line segment,
$\therefore \overrightarrow{AG}\overrightarrow{AC}\overrightarrow{AJ}$==,
Lemma 4: For any angle arc, the segment obtained by iteratively adding one-third of the sum of the three sides of the trapezoid's upper base is precisely equal to the target segment, which is the one-third chord of the arc.
Yes, the midline.$\because \overrightarrow{AC}=\overrightarrow{AJ}$, $\overrightarrow{AYD}\overrightarrow{AJ}$
$\therefore \overrightarrow{AYD}\perp\overrightarrow{CJ}\overrightarrow{CY}=\overrightarrow{JY}$
$\because \overrightarrow{OA}=\overrightarrow{OD}\overrightarrow{OY}\perp\overrightarrow{AYD}$
$\therefore \overrightarrow{AY}=\overrightarrow{YD}$
$\because \overrightarrow{AD}\perp\overrightarrow{CJ}\ \overrightarrow{AY}=\overrightarrow{YD}\overrightarrow{CY}=\overrightarrow{YJ}$
$\therefore\ \angle DAJ=\angle CDA,\angle DCJ=\angle AJC$
$\because \angle DAJ=\angle CDA,\angle DCJ=\angle AJC\ \overrightarrow{AY}=\overrightarrow{YD}\overrightarrow{CY}=\overrightarrow{YJ}$
$\therefore \overrightarrow{AJ}=\overrightarrow{CD}$,
$\because \angle DAJ=\angle CDA,\angle DCJ=\angle AJC$,
$\therefore \overrightarrow{CD}\parallel\overrightarrow{AB}$
$\because \overrightarrow{CD}\parallel\overrightarrow{AB}\widehat{AB}$
Both point C and point D are located above.
$\therefore \overrightarrow{KA}+\overrightarrow{AH}+\overrightarrow{KA},\overrightarrow{AC}=\overrightarrow{DB}$,
$\therefore \overrightarrow{AC}=\overrightarrow{CD}=\overrightarrow{BD}$

$\because \overrightarrow{AC} = \overrightarrow{CD} = \overrightarrow{DB}\overrightarrow{OA} = \overrightarrow{OC} = \overrightarrow{OD} = \overrightarrow{OB}$

$\therefore \angle AOC = \angle COD, \angle DOB = \angle AOB/3$

Substitute for numerical verification: (Assume)

$\overrightarrow{OA} = \overrightarrow{OB} = \sqrt{5AB} = 4.4, \overrightarrow{AH} = \overrightarrow{HB} = 2.2, \overrightarrow{OH} = 0.4$

After calculation, the $\overrightarrow{AG_1}\overrightarrow{Z_1H}\overrightarrow{AG_1}$ constructed line segment is 1.9986..., and /2 is 0.999345...

After $\overrightarrow{AC_1}\overrightarrow{AG_2}\ \overrightarrow{Z_2H}$the first iteration, the value is 2.00032...; after the second iteration, it is 1.99999996...; and after the third iteration, it is 0.99999998...

The $\overrightarrow{AC_2}\overrightarrow{AC}\overrightarrow{AG}\overrightarrow{AJ} = 2\overrightarrow{ZH}$ result of the first iteration is 2.00000002...

The result of the second iteration is ==,=1

2.2 The second case

$Given: \angle AOB\overrightarrow{OA} = \overrightarrow{OB}(small\ obtuse\ angle),$

$\overrightarrow{AH} = \overrightarrow{BH} = \overrightarrow{AB}/2$

Request: $\angle AOC = \angle COD = \angle DOB = \angle AOB/3$

Drawing: (see Figure 2)

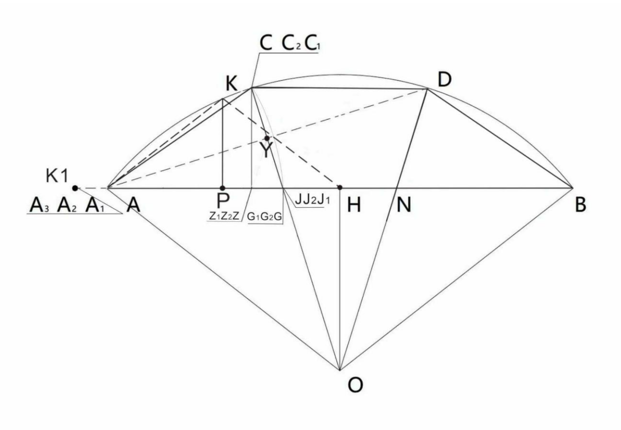

Figure 2. Proof of Angle Trisection.

1. $\angle AOB, O\overrightarrow{OA}\widehat{AB}\overrightarrow{AB}$

Draw a circle with the point as the center and the radius as the length, then connect the points.

2. Make a deal. $\overrightarrow{OH} \perp \overrightarrow{AB}\overrightarrow{AB}$ YuDotH,

3. Draw $\overrightarrow{AH}$, the perpendicular bisectors of $\overrightarrow{AH}\widehat{AB}$ a line segment intersecting point P and point K, forming an isosceles triangle KAH.

4. Draw an P$\overrightarrow{AK}\overrightarrow{HA}\overrightarrow{K_1}$ arc with a point as the center and a radius, then extend the line of intersection to point

5. Three equal parts, $=/2\overrightarrow{K_1H}\overrightarrow{K_1Z_1} = (\overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH})/32\overrightarrow{K_1H}/3$, $\overrightarrow{Z_1H}\overrightarrow{K_1H}/3\overrightarrow{K_1Z_1}$

6. The line $Z_1\overrightarrow{C_1Z_1} \perp \overrightarrow{AH}\widehat{AB}C_1$ passing through the point intersects at another point.

7. Draw an $Z_1A\ \overrightarrow{C_1H}\overrightarrow{AA_1}\overrightarrow{A_1H}\overrightarrow{Z_2H} = \frac{\overrightarrow{A_1H}}{3}$arc from a point as the center and radius, intersect the extension line at that point, then divide it into three equal parts.

$\overrightarrow{A_1Z_2} = \frac{2\overrightarrow{A_1H}}{3} = 2\overrightarrow{Z_2H}$

8. The line $Z_2\overrightarrow{C_2Z_2} \perp \overrightarrow{AH}\widehat{AB}C_2$ passing through the point intersects at another point.

9. Draw $Z_2A\ \overrightarrow{C_2H}\overrightarrow{AA_2}\overrightarrow{A_2H}\overrightarrow{ZH} = \overrightarrow{A_2H}/3 = \overrightarrow{A_2Z}/2$ an arc from the center to the point where the extension of the radius intersects, and divide it into three equal parts.

10. Draw an arc from $\overrightarrow{A_2Z}AB$ intersects at point $C\overrightarrow{AH}$ point A with a radius equal to the distance from A to point J, intersecting point J.

11. The connecting $\overrightarrow{CO}\overrightarrow{AH}\angle AOC = \angle\frac{AOB}{3}$ line coincides with the intersection point J.

Prove: $\angle AOC = \angle AOB/3$

Create a guideline:

Draw an arc from point A with radius equal to 2, intersecting at point.

$$\overrightarrow{K_1Z_1}\overrightarrow{AH}G_1\ \overrightarrow{AG_1}\overrightarrow{K_1Z_1} = \overrightarrow{Z_1H}$$

Draw an arc from $\overrightarrow{A\,C_1}\overrightarrow{AH}J_1\ \overrightarrow{A\,C_1}\overrightarrow{\mathrm{A}\,\mathrm{J}_1}$point A with radius equal to the distance from point A to the intersection point.

Draw an arc from point A with radius equal to 2, intersecting at point.

$$\overrightarrow{A_1Z_2}\overrightarrow{AH}G_2\overrightarrow{AG_2}\overrightarrow{A_1Z_Z} = \overrightarrow{Z_2H}$$

Draw an arc from $\overrightarrow{A\,C_2}\overrightarrow{AH}J_2\overrightarrow{A\,C_2}\overrightarrow{A\,J_2}$ point A with radius equal to the distance from point A to the intersection point.

The $\overrightarrow{CJ}\overrightarrow{CJ}$ at the point Y, iao$\widehat{AB}$perpendicular bisectors of the triangle intersect at point D.

The connecting $\overrightarrow{OD}\overrightarrow{HB}\overrightarrow{CD}\ \overrightarrow{DB}$lines intersect at point N.

Prove:

$\angle AOC = \angle COD = \angle DOB = \angle AOB/3\ =$

$\because \overrightarrow{AP}\overrightarrow{PH}\frac{\overrightarrow{AH}}{2}, \overrightarrow{KA} = \overrightarrow{KH}, \overrightarrow{K_1P} = \overrightarrow{KA}$

$\therefore (\overrightarrow{KA} + \overrightarrow{KP}\overrightarrow{AH}\overrightarrow{KA} + \overrightarrow{AP}+)/2 =,$

$\because \overrightarrow{K_1P} = \overrightarrow{AK},$

$\therefore\ (\overrightarrow{AK} + \overrightarrow{PH}\overrightarrow{K_1P} + \overrightarrow{PH})\ /3,$

$\because \overrightarrow{K_1Z_1} = 2\ \overrightarrow{K_1H}\overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH}\overrightarrow{K_1H}\ \overrightarrow{K_1Z_1} = \overrightarrow{AG_1}$

$\therefore \overrightarrow{AG_1} = \frac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}$

AG_1 is the target line segment, and AG_1 is less than a few thousandths of the target line segment.

Proposition 2: For a trapezoid with unequal sides on any angle segment, the construction line segment equals one-third of the sum of its three sides, and this construction line segment is less than a few thousandths of the target line segment.

$\because \overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH}$

In an isosceles triangle KAH, the three sides of the trapezoid formed by the top base of the arc trapezoid constructed on the base of the isosceles triangle are equal to

$\therefore \overrightarrow{AG_1} = \frac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}$

The requested constructed line segment

$\because \overrightarrow{AG_1}\overrightarrow{K_1Z_1}\overrightarrow{Z_1H}\overrightarrow{AG_1}$+)/3,

$\therefore$ One iteration of the trapezoid′s three sides sum divided by three$=2(\overrightarrow{A_1Z_1}\overrightarrow{Z_1H}\overrightarrow{AG_2},$

$\because \overrightarrow{AG_2}\overrightarrow{AG_1}$

It constructs a segment equal to one-third of the sum of the three sides of the trapezoid with the upper base as its base, which is a one-time iteration segment.

$\therefore \overrightarrow{AG_2}\overrightarrow{AG_1}$

As stated in Lemma 3, the construction line segment on any angle arc is one-third of the sum of the three sides of the upper trapezoid, which constitutes the iteration line segment. This iteration line segment is between a few millionths and a few hundredths of the target line segment. As stated in Lemma 3, the construction line segment on any angle arc is one-third of the sum of the three sides of the upper trapezoid, which constitutes the iteration line segment. This iteration line segment is between a few millionths and a few hundredths of the target line segment.

=2/3,=/3=/2,=,$\because \overrightarrow{A_1Z_2}\overrightarrow{A_1H}\ \overrightarrow{Z_2H}\overrightarrow{A_1H}\overrightarrow{A_1Z_2}\ \overrightarrow{A_1Z_2}\overrightarrow{AG_2}$/3=+)/3,

$\therefore \overrightarrow{AG_2} = 2\overrightarrow{A_1H}2(\overrightarrow{AC_1}\overrightarrow{Z_1H}$

Greater than or equal to /3=+) / 3, +) /3,

$\because \overrightarrow{AC_2}\overrightarrow{AC_1}, \overrightarrow{AG_2}2\overrightarrow{A_1H}2(\overrightarrow{AC_2}\overrightarrow{Z_2H}\overrightarrow{AG_1} = 2(\overrightarrow{AC_1}\overrightarrow{Z_1H}$+) / 3 is greater than +) / 3,

$\therefore 2(\overrightarrow{AC_2}\overrightarrow{Z_2H}2(\overrightarrow{AC_1}\overrightarrow{Z_1H}$

$\because \overrightarrow{AG}2(\overrightarrow{AC_2}\overrightarrow{Z_2H}\ \overrightarrow{AG}\overrightarrow{AG}\overrightarrow{AC}$=+) / 3 is the second iteration line segment, =,

$\therefore \overrightarrow{AG}\overrightarrow{AC}\overrightarrow{AJ}$==,

Lemma 4: For any angle arc, the segment obtained by iteratively adding one-third of the sum of the three sides of the trapezoid's upper base is precisely equal to the target segment, which is the one-third chord of the arc.

Yes, the midline.$\because \overrightarrow{AC}=\overrightarrow{AJ}, \overrightarrow{AYD}\overrightarrow{AJ}$

$\therefore \overrightarrow{AYD} \perp \overrightarrow{CJ}\overrightarrow{CY} = \overrightarrow{JY}$,

$\because \overrightarrow{OA} = \overrightarrow{OD}\overrightarrow{OY} \perp \overrightarrow{AYD}$

$\therefore \overrightarrow{AY} = \overrightarrow{YD}$

$\because \overrightarrow{AD} \perp \overrightarrow{CJ}\ \overrightarrow{AY} = \overrightarrow{YD}\overrightarrow{CY} = \overrightarrow{YJ}$

$\therefore\ \angle DAJ = \angle CDA, \angle DCJ = \angle AJC$

$\because \angle DAJ = \angle CDA, \angle DCJ = \angle AJC\ \overrightarrow{AY} = \overrightarrow{YD}\overrightarrow{CY} = \overrightarrow{YJ}$

$\therefore \overrightarrow{AJ} = \overrightarrow{CD}$

$\because \angle DAJ = \angle CDA, \angle DCJ = \angle AJC$

$\therefore \overrightarrow{CD} \parallel \overrightarrow{AB}$

$\because \overrightarrow{CD} \parallel \overrightarrow{AB}\widehat{AB}$

Both points C and D are located above.

$\therefore$ Point C is symmetric to point D, and AC is symmetric to DB.

$\overrightarrow{AC}=\overrightarrow{DB}$,

$\therefore \overrightarrow{AC} = \overrightarrow{CD} = \overrightarrow{BD}$

$\because \overrightarrow{AC} = \overrightarrow{CD} = \overrightarrow{DB}\overrightarrow{OA} = \overrightarrow{OC} = \overrightarrow{OD} = \overrightarrow{OB}$

$\therefore \angle AOC = \angle COD, \angle DOB = \angle AOB/3$

Substitute the numbers for verification:

let, $\overrightarrow{AB} = 5.2\overrightarrow{AH}=\overrightarrow{BH}=2.6$, $\overrightarrow{OH} = 1.8$

$=3.16227766016838\overrightarrow{OA} = \overrightarrow{OB} = \sqrt{10}$

The calculated constructed $\overrightarrow{AG_1}\overrightarrow{Z_1}\overrightarrow{H}\overrightarrow{AG_1}$line segment is 1.99452..., and the square root of 2 is 0.99726...

After $\overrightarrow{AC_1}\overrightarrow{AG_2}\ \overrightarrow{Z_2H}$the first iteration, the value is 2.00272...;

The subsequent iterations yield 1.99999958... and 0.99999979...

The $\overrightarrow{AC_2}\overrightarrow{AC}\overrightarrow{AG}\overrightarrow{AJ} = 2\overrightarrow{ZH}$ result of the first iteration is 2.000000296..., and the result of the second iteration is 1.

2.3 The third case

Known: $\angle AB = 90^0$(right angle)

$\overrightarrow{OA} = \overrightarrow{OB}$, $\overrightarrow{AH} = \overrightarrow{BH} = \overrightarrow{OH} = \overrightarrow{AB}/2$

Request: $\angle AOC = \angle COD = \angle DOB = \angle AOB/3$

Drawing: (see Figure 3)

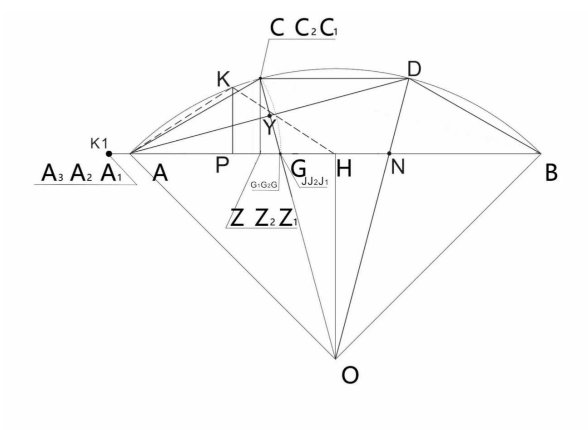

Figure 3. Proof of Angle Trisection.

1. Passing through point H1, in $\overrightarrow{AB}$ Do something online H, $\overrightarrow{AB}$ Perpendicular line $\overrightarrow{OH}$.

2. Draw an arc with point H as the center and $\overrightarrow{OH}$ Dot O, Let AB intersect at point A any length as the radius, intersecting point B.

Draw a circle with $\overrightarrow{OA}\widehat{AB}$ point O as the center and the radius as

3. Draw $\overrightarrow{AH}$, the perpendicular bisectors of $\overrightarrow{AH}\widehat{AB}$ a line segment intersecting point P and point K, forming an isosceles triangle KAH.

4. Draw an P$\overrightarrow{AK}\overrightarrow{HA}K_1$ arc with a point as the center and a radius, then extend the line of intersection to point

5. Three equal parts, $=/2\overrightarrow{K_1H}\overrightarrow{K_1Z_1} = (\overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH})\ /32\overrightarrow{K_1H}/3, \overrightarrow{Z_1H}\overrightarrow{K_1H}/3\overrightarrow{K_1Z_1}$

6. The line $Z_1\overrightarrow{C_1Z_1} \perp \overrightarrow{AH}\widehat{AB}C_1$ passing through the point intersects at another point.

7. Draw an $Z_1\overrightarrow{A\ C_1}\overrightarrow{HA}A_1\overrightarrow{A_1H}\overrightarrow{Z_2H} = \frac{\overrightarrow{A_1H}}{3}$ arc from a point as the center and radius, intersect the extension line at that point, then divide it into three equal parts.

$\overrightarrow{A_1Z_2} = \frac{2\overrightarrow{A_1H}}{3} = 2\overrightarrow{Z_2H}$

8. The line $Z_2\overrightarrow{C_2Z_2} \perp \overrightarrow{AH}\widehat{AB}C_2$ passing through the point intersects at another point.

9. Draw $Z_2\overrightarrow{A\ C_2}\overrightarrow{HA}A_2\overrightarrow{A_2H}\overrightarrow{ZH} = \overrightarrow{A_2H}/3 = \overrightarrow{A_2Z}/2$ an arc from the center to the point where the extension of the radius intersects, and divide it into three equal parts.

10. Draw an arc from $\overrightarrow{A_2Z}\widehat{AB}$ at point C, intersect $\overrightarrow{AH}$ point A with a radius equal to the distance from A to point J, intersecting point J.

11. The connecting $\overrightarrow{CO}\overrightarrow{AH}\angle AOC = \angle\frac{AOB}{3}$ line coincides with the intersection point J.

Prove: $\angle AOC = \angle AOB/3$

Create a guideline:

Draw an arc from point A with radius equal to 2, intersecting at point.

$\overrightarrow{K_1Z_1}\overrightarrow{AH}G_1\ \overrightarrow{AG_1}\overrightarrow{K_1Z_1} = \overrightarrow{Z_1H}$

Draw an arc from $\overrightarrow{A\ C_1}\overrightarrow{AH}J_1\ \overrightarrow{A\ C_1}\overrightarrow{A\ J_1}$ point A with radius equal to the distance from point A to the intersection poin t.

Draw an arc from point A with radius equal to 2, intersecting at point.

$\overrightarrow{A_1Z_2}\overrightarrow{AH}G_2\overrightarrow{AG_2}\overrightarrow{A_1Z_Z} = \overrightarrow{Z_2H}$

Draw an arc from $\overrightarrow{A\ C_2}\overrightarrow{AH}J_2\overrightarrow{A\ C_2}\overrightarrow{A\ J_2}$ point A with radius equal to the distance from point A to the intersection point.

The $\overrightarrow{CJ}\overrightarrow{CJ}$ at the point Y, iao$\widehat{AB}$ perpendicular bisectors of the triangle intersect at point D.

The connecting $\overrightarrow{OD}\overrightarrow{HB}\overrightarrow{CD}\ \overrightarrow{DB}$ lines intersect at point N.

Prove:

$\angle AOC = \angle COD = \angle DOB = \angle AOB/3$

$\because \overrightarrow{AP}\overrightarrow{PH}\frac{\overrightarrow{AH}}{2}, \overrightarrow{KA} = \overrightarrow{KH},\ \overrightarrow{K_1P} = \overrightarrow{KA},$

$\therefore$ （$\overrightarrow{KA} + \overrightarrow{KP} + \overrightarrow{AH}$） $= \overrightarrow{KA} + \overrightarrow{AP}$）

$\because \overrightarrow{K_1P} = \overrightarrow{AK},$

$\therefore \overrightarrow{AK} + \overrightarrow{PH}\overrightarrow{K_1P} + \overrightarrow{PH},$

$\because \overrightarrow{K_1Z_1} = 2\ \overrightarrow{K_1H}\overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH}\overrightarrow{K_1H}\ \overrightarrow{K_1Z_1} = \overrightarrow{AG_1},$

$\therefore \overrightarrow{AG_1} = \frac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}, \overrightarrow{AG_1}$

For the target line segment,

Lemma 2: For a trapezoid with unequal sides on any angle segment, one-third of the sum of its three sides can be constructed as a line segment, which is a fraction of the target line segment.

$\because \overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH}$

In an isosceles triangle KAH, the three sides of the trapezoid formed by the top base of the arc trapezoid constructed on the base of the isosceles triangle are equal to

$\therefore \overrightarrow{AG_1} = \frac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}$

The requested constructed line segment,

$\because \overrightarrow{AG_1}_{=}\overrightarrow{K_1Z_1}, \overrightarrow{Z_1H} = \overrightarrow{AG_1}/2$

$\therefore$ One iteration of one third of the sum of the three sides of a trapezoid$=2(\overrightarrow{A_1Z_1}+\overrightarrow{Z_1H} = \overrightarrow{AG_2}/3$

$\because \overrightarrow{AG_2}\overrightarrow{AG_1}$

It constructs a segment equal to one-third of the sum of the three sides of the trapezoid with the upper base as its base, which is a one-time iteration segment.

$\therefore \overrightarrow{AG_2}\overrightarrow{AG_1}$

As stated in Lemma 3, the construction line segment on any angle arc is one-third of the sum of the three sides of the upper trapezoid, which constitutes the iteration line segment. This iteration line segment is between a few millionths and a few hundredths of the target line segment.

=2/3,=/3=/2,=,$\because \overrightarrow{A_1Z_2}\overrightarrow{A_1H}\ \overrightarrow{Z_2H}\overrightarrow{A_1H}\overrightarrow{A_1Z_2}\ \overrightarrow{A_1Z_2}\overrightarrow{AG_2}$/3=+)/3,

$\therefore \overrightarrow{AG_2}=2\overrightarrow{A_1H}2(\overrightarrow{AC_1}\overrightarrow{Z_1H}$

Greater than or equal to

/3=+)/3, +) /3,$\because \overrightarrow{AC_2}\overrightarrow{AC_1}, \overrightarrow{AG_2}2\overrightarrow{A_1H}2(\overrightarrow{AC_2}\overrightarrow{Z_2H}\overrightarrow{AG_1}=2(\overrightarrow{AC_1}\overrightarrow{Z_1H}$+)/3 is greater than +)/3,

$\therefore 2(\overrightarrow{AC_2}\overrightarrow{Z_2H}2(\overrightarrow{AC_1}\overrightarrow{Z_1H}$

$\because \overrightarrow{AG}2(\overrightarrow{AC_2}\overrightarrow{Z_2H}\ \overrightarrow{AG}\overrightarrow{AG}\overrightarrow{AC}$=+) / 3 is the second iteration line segment,

$\therefore \overrightarrow{AG}\overrightarrow{AC}\overrightarrow{AJ}$=,

Lemma 4: For any angle arc, the segment obtained by iteratively adding one-third of the sum of the three sides of the trapezoid's upper base is precisely equal to the target segment, which is the one-third chord of the arc.

Yes, the midline.

$\because \overrightarrow{AC}=\overrightarrow{AJ}, \overrightarrow{AYD} \perp \overrightarrow{AJ},$

$\therefore \overrightarrow{AYD} \perp \overrightarrow{CJ}\overrightarrow{CY} = \overrightarrow{JY}$

$\because \overrightarrow{OA} = \overrightarrow{OD}\overrightarrow{OY} \perp \overrightarrow{AYD}$

$\therefore \overrightarrow{AY} = \overrightarrow{YD}$

$\because \overrightarrow{AD} \perp \overrightarrow{CJ}\ \overrightarrow{AY} = \overrightarrow{YD}\overrightarrow{CY} = \overrightarrow{YJ}$

$\therefore\ \angle DAJ = \angle CDA, \angle DCJ = \angle AJC$

$\because \angle DAJ = \angle CDA, \angle DCJ = \angle AJC\ \overrightarrow{AY} = \overrightarrow{YD}\overrightarrow{CY} = \overrightarrow{YJ}$

$\therefore \overrightarrow{AJ} = \overrightarrow{CD}$

$\because \angle DAJ = \angle CDA, \angle DCJ = \angle AJC$

$\therefore \overrightarrow{CD} \parallel \overrightarrow{AB}$

$\because \overrightarrow{CD} \parallel \overrightarrow{AB}\widehat{AB}$

Both points C and D are located above.

$\therefore$ Point C and point D Symmetry, $AC\ is\ symmetric\ to\ DB, \overrightarrow{AC}=\overrightarrow{DB},$

$\therefore \overrightarrow{AC} = \overrightarrow{CD} = \overrightarrow{BD}$

$\because \overrightarrow{AC} = \overrightarrow{CD} = \overrightarrow{DB}\overrightarrow{OA} = \overrightarrow{OC} = \overrightarrow{OD} = \overrightarrow{OB}$

$\therefore \angle AOC = \angle COD, \angle DOB = \angle AOB/3$

Substitute the numbers for verification:

set, /2$\overrightarrow{AB} = 5.46410161513776\ \overrightarrow{AH} = \overrightarrow{BH} = \overrightarrow{OH} = \overrightarrow{AB}$

$\overrightarrow{OA} = \overrightarrow{OB} = \sqrt{5.46410161513776}$

After calculation, the $\overrightarrow{AG_1}\overrightarrow{Z_1H}\overrightarrow{AG_1}$ constructed line segment is 1.99452..., and the ratio of 1 to 2 is 0.99736...

After $\overrightarrow{AC_1}\overrightarrow{AG_2}\ \overrightarrow{Z_2H}$ the first iteration, the value is 2.00262...; after the second iteration, it is 1.99999974...; and after the third iteration, it is 0.99999987...

The $\overrightarrow{AC_2}\overrightarrow{AC}\overrightarrow{AG}\overrightarrow{AJ} = 2\overrightarrow{ZH}$ result of the first iteration is 2.000000124..., and the result of the second iteration is 1.

2.4 The fourth case

Given: (acute angle) $\angle AOB = 60^0, \overrightarrow{AB} = \overrightarrow{OA} = \overrightarrow{OB}\ \overrightarrow{AH} = \overrightarrow{AB}/2,$

Request: $\angle AOC = \angle \mathrm{COD} = \angle \mathrm{DOB} = \frac{\angle AO\mathrm{B}}{3}$,

Drawing: (see Figure 4)

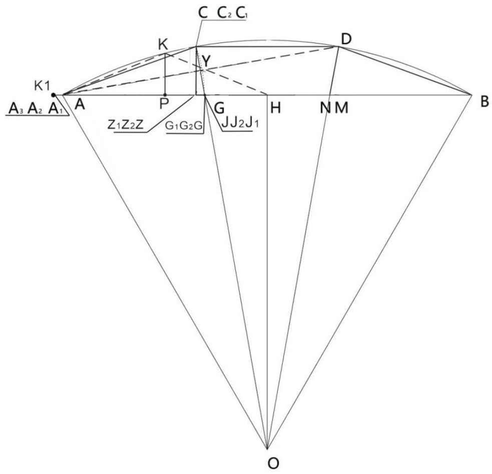

Figure 4. Proof of Angle Trisection.

1. Draw $OO\overrightarrow{A}\widehat{AB}\widehat{AB}\mathrm{A}\overrightarrow{OA}\widehat{AB}$ at the point B, a circle with the point as the center and the radius as the length, then draw an arc with the same point as the center and intersect it.

Linkage, $\overrightarrow{OA} = \overrightarrow{OB}$

2. Make a deal. $\overrightarrow{OH} \perp \overrightarrow{AB}\overrightarrow{AB}$ at point H,

3. Draw the perpendicular bisectors of AH respectively $\overrightarrow{AH}\widehat{AB}$, a line segment intersecting point P and point K, forming an isosceles triangle KAH.

4. Draw an $\mathrm{P}\overrightarrow{AK}\overrightarrow{HA}K_1$ arc with a point as the center and a radius, then extend the line of intersection to point

5. Three equal parts, $=/2\overrightarrow{K_1H}\overrightarrow{K_1Z_1} = (\overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH})\ /32\overrightarrow{K_1H}/3, \overrightarrow{Z_1H}\overrightarrow{K_1H}/3\overrightarrow{K_1Z_1}$

6. The line $Z_1\overrightarrow{C_1Z_1} \perp \overrightarrow{AH}\widehat{AB}C_1$ passing through the point intersects at another point.

7. Draw an $Z_1\overrightarrow{A\ C_1HA}\overrightarrow{A_1A_1H}\overrightarrow{Z_2H} = \overrightarrow{A_1H}/3$ arc from a point as the center and radius, intersect the extension line at that point, then divide it into three equal parts,

$\overrightarrow{A_1Z_2} = \frac{2\overrightarrow{A_1H}}{3} = 2\overrightarrow{Z_2H}$

8. The line $Z_2\overrightarrow{C_2Z_2} \perp \overrightarrow{AH}\widehat{AB}C_2$ passing through the point intersects at another point.

9. Draw $Z_2\overrightarrow{A\ C_2HA}\overrightarrow{A_2A_2H}\overrightarrow{ZH} = \overrightarrow{A_2H}/3 = \overrightarrow{A_2Z}/2$ an arc from the center to the point where the extension of the radius intersects, and divide it into three equal parts.

10. Draw an arc from $\overrightarrow{A_2Z}AB$ intersects AH at point C, point A with a radius equal to the distance from A to point J, intersecting point J.

11. The connecting $\overrightarrow{CO}\overrightarrow{AH}\angle AOC = \angle\frac{AOB}{3}$ line coincides with the intersection point J.

Prove: $\angle AOC = \angle AOB/3$,

Create a guideline:

Draw an arc from point A with radius equal to 2, intersecting at point.

$\overrightarrow{K_1Z_1} = \overrightarrow{AG_1}, \overrightarrow{AG_{1=}K_1Z_1} = 2\overrightarrow{Z_1H}$

Draw an arc from $\overrightarrow{A\ C_1AH}J_1\ \overrightarrow{A\ C_1A\ J_1}$ point A with radius equal to the distance from point A to the intersection point.

Draw an arc from point A with radius equal to 2, intersecting at point.

$\overrightarrow{A_1Z_2AH}G_2\overrightarrow{AG_2A_1Z_Z} = \overrightarrow{Z_2H}$

Draw an arc from $\overrightarrow{A\ C_2AH}J_2\overrightarrow{A\ C_2A\ J_2}$ point A with radius equal to the distance from point A to the intersection point.

The $\overrightarrow{CJ}\overrightarrow{CJ}$ at the point Y, iao$\widehat{AB}$ perpendicular bisectors of the triangle intersect at point D.

The connecting $\overrightarrow{OD}\overrightarrow{HB}\overrightarrow{CD}\ \overrightarrow{DB}$ lines intersect at point N.
Prove: $\angle AOC = \angle COD = \angle DOB = \angle AOB/3$
$\because \overrightarrow{AP}\overrightarrow{PH}\frac{\overrightarrow{AH}}{2}, \overrightarrow{KA} = \overrightarrow{KH}, \overrightarrow{K_1P} = \overrightarrow{KA},$
$\therefore$ （$\overrightarrow{KA} + \overrightarrow{KP} + \overrightarrow{AH}$） $= 2$ （$\overrightarrow{KA} + \overrightarrow{AP}$）
$\because \overrightarrow{K_1Z} = \overrightarrow{AK}$
$\therefore \overrightarrow{AK} + \overrightarrow{PH} = \overrightarrow{K_1P} + \overrightarrow{PH},$
$\because \overrightarrow{K_1Z_1}=2\ \overrightarrow{K_1H}\overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH}\overrightarrow{K_1H}\ \overrightarrow{K_1Z_1}=\overrightarrow{AG_1},$
$\therefore \overrightarrow{AG_1} = \frac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}, \overrightarrow{AG_1}$
For the target line segment,
$\because \overrightarrow{KA} + \overrightarrow{KH} + \overrightarrow{AH}$
In an isosceles triangle KAH, the three sides of the trapezoid formed by the top base of the arc trapezoid constructed on the base of the isosceles triangle are equal to
$\therefore \overrightarrow{AG_1} = \frac{\overrightarrow{KA}+\overrightarrow{KH}+\overrightarrow{AH}}{3}$
The constructed line segment is the one that satisfies the condition: according to Lemma 2, the sum of the three sides of an isosceles trapezoid with unequal sides on any angle arc equals one-third of the constructed line segment. The constructed line segment is a fraction of the target line segment.
$\because \overrightarrow{AG_{1=}}\overrightarrow{K_1Z_{1,,}}\overrightarrow{Z_1H} = \overrightarrow{AG_1}/2,$
$\therefore$ One iteration of the trapezoid′s three sides sum divided by three$=2(\overrightarrow{A_1Z_1}\overrightarrow{Z_1H}\overrightarrow{AG_2},$
$\because \overrightarrow{AG_2}\overrightarrow{AG_1}$
It constructs a segment equal to one-third of the sum of the three sides of the trapezoid with the upper base as its base, which is a one-time iteration segment.
$\therefore \overrightarrow{AG_2}\overrightarrow{AG_1}$
As stated in Lemma 3, the construction line segment on any angle arc is one-third of the sum of the three sides of the upper trapezoid, which constitutes the iteration line segment. This iteration line segment is between a few millionths and a few hundredths of the target line segment. As stated in Lemma 3, the construction line segment on any angle arc is one-third of the sum of the three sides of the upper trapezoid, which constitutes the iteration line segment. This iteration line segment is between a few millionths and a few hundredths of the target line segment.
$\because \overrightarrow{A_1Z_2} = \overrightarrow{2A_1H/3}, \overrightarrow{Z_2H} = \overrightarrow{A_1H/3,,A_1Z_2} = \overrightarrow{AG_2},$
$\therefore \overrightarrow{AG_2}=2\overrightarrow{A_1H}/3,$
$\because \overrightarrow{AC_{2=}}\overrightarrow{AC_1}$, $\overrightarrow{AG_2} = \overrightarrow{2Z_1H},,$
$,\therefore \overrightarrow{AC_{2=}}2\overrightarrow{Z_2H}, \overrightarrow{AG_{1=}}2\overrightarrow{Z_2H}$
$\because \overrightarrow{AG}2(\overrightarrow{AC_2}\overrightarrow{Z_2H}\ \overrightarrow{AG}\overrightarrow{AG}\overrightarrow{AC}$=+) / 3 is the second iteration line segment, =,
$\therefore \overrightarrow{AG}\overrightarrow{AC}\overrightarrow{AJ}$==,
Lemma 4: For any angle arc, the segment obtained by iteratively adding one-third of the sum of the three sides of the trapezoid's upper base is precisely equal to the target segment, which is the one-third chord of the arc.
Yes, the midline.
$\because \overrightarrow{AC}=\overrightarrow{AJ}$, $\overrightarrow{AYD}\overrightarrow{AJ}$
$\therefore \overrightarrow{AYD} \perp \overrightarrow{CJ}\overrightarrow{CY} = \overrightarrow{JY}$
$\because \overrightarrow{OA} = \overrightarrow{OD}\overrightarrow{OY} \perp \overrightarrow{AYD}$
$\therefore \overrightarrow{AY} = \overrightarrow{YD}$
$\because \overrightarrow{AD} \perp \overrightarrow{CJ}\ \overrightarrow{AY} = \overrightarrow{YD}\overrightarrow{CY} = \overrightarrow{YJ}$
$\therefore\ \angle DAJ = \angle CDA$, $\angle DCJ = \angle AJC$
$\because \angle DAJ = \angle CDA$, $\angle DCJ = \angle AJC\ \overrightarrow{AY} = \overrightarrow{YD}\overrightarrow{CY} = \overrightarrow{YJ}$
$\therefore \overrightarrow{AJ} = \overrightarrow{CD}$
$\because \angle DAJ = \angle CDA$, $\angle DCJ = \angle AJC,$
$,\therefore \overrightarrow{CD} \parallel \overrightarrow{AB}$

$\because \overrightarrow{CD} \parallel \overrightarrow{AB}\widehat{AB}$

Both points C and D are located above.

$\therefore$ Point C is symmetric to Point DAC is symmetric with DB, $\overrightarrow{AC}=\overrightarrow{DB}$,

,$\therefore \overrightarrow{AC} = \overrightarrow{CD} = \overrightarrow{BD}$

$\because \overrightarrow{AC} = \overrightarrow{CD} = \overrightarrow{DB}\overrightarrow{OA} = \overrightarrow{OC} = \overrightarrow{OD} = \overrightarrow{OB}$,

$\therefore \angle AOC = \angle COD, \angle DOB = \angle AOB/3$

Substitute the numbers for verification:

Assume $\overrightarrow{AB} = \overrightarrow{OA} = \overrightarrow{OB} = 5.75877048314364/2 = \overrightarrow{AH} = \overrightarrow{BH} = \overrightarrow{OH} = \overrightarrow{AB}$ 2.87938524157182, $\overrightarrow{OH} =$ 4.98724153296638

The calculated constructed $\overrightarrow{AG_1}\overrightarrow{Z_1H}\overrightarrow{AG_1}$ line segment is 1.9962..., and the square root of 2 is 0.9981...

After $\overrightarrow{AC_1}\overrightarrow{AG_2}\ \overrightarrow{Z_2H}$ the first iteration, the value is 2.00162, followed by 1.999999916 and 0.999999958.

The $\overrightarrow{AC_2}\overrightarrow{AC}\overrightarrow{AG}\overrightarrow{AJ} = 2\overrightarrow{ZH}$ result of the first iteration is 2.00000002...

The result of the second iteration is 1

3. Conclusion

The proposition of trisecting an angle is a classic problem with no solution. After acknowledging that compass and straightedge cannot directly produce the trisecting points of arcs corresponding to arbitrary angles, the insistence on trisecting arbitrary angles remains an impossible proposition. This stems from the belief that trisecting a 180-degree angle involves constructing a line segment equal to one-third of the chord length corresponding to the arc. If such a segment could be drawn, trisecting arbitrary angles might become feasible. However, the one-third chord of an arc originates from its trisecting points—how could one draw such a chord without these points? Thus, researchers explored the geometric variations of constructing trapezoids with equal or unequal sides corresponding to arcs of arbitrary angles. Over six decades of exploration, countless hypotheses, conjectures, and attempts were made. Eventually, a method was discovered: using a trapezoid with unequal top base and legs to achieve equal sides. Initial attempts to construct seemingly equal triangles were later disproven by calculations, leading to further analysis confirming the method's validity—though not yet perfected. (The original construction used a trapezoid with the top base as the side, employing a method that subtracted two-thirds of the difference between the top base and legs.) Although recognizing that repeating this method could reduce the top base-leg difference, the infinitesimal difference remained undetectable. It took over a decade to discover that trisecting the sum of three sides yields the same result. Only after reaching eighty years old did the current construction method emerge, with calculations showing that the top base-leg difference transitions from 15-digit numbers to special cases of integer values, ultimately achieving trapezoids with integer side lengths corresponding to all arcs of arbitrary angles. However, all domestic submissions were rejected. To address concerns about universality and precision, a second paper was published [2]. Since both proofs relied on computational results and did not meet the rigorous standards of geometric proofs, this paper is now presented to provide a strict geometric proof, demonstrating that the proposed graphical method satisfies the requirements of geometric proof.

Previous studies had utilized the rule of trapezoidal side changes on arbitrary angle arcs to construct angle-trisection diagrams, yet failed to elevate this principle to theoretical rigor for formal proof—the lemma proposed in this paper's introduction. Earlier constructions relied on the trapezoid's base being an integer multiple of the leg length, but could not yield a rigorous geometric proof. While this paper's lemma meets theoretical proof requirements, the introductory conventions alone are insufficient. Thus, this section provides additional examples to substantiate the lemma: the isosceles trapezoid $\sqrt{0.84}\sqrt{0.84+0.36}$ property on arbitrary angle arcs requires no further proof, whereas the trapezoidal sum theorem (equal legs) surpasses the unequal-leg case through multiple variations. For instance, in a right-angled isosceles trapezoid, the base equals half the hypotenuse. If we assume the base equals 0.4 times the hypotenuse, the trapezoid's height equals, and the legs equal=

$$\sqrt{1.2}\sqrt{0.751517}\sqrt{0.751517+0.2515223}\sqrt{1.0030393}$$

The value 1.09544511501033 corresponds to a geometric ratio where the sum of three sides equals 2.99089... One-third of the constructed line segment measures 0.99696..., while the target line segment is approximately 0.03% smaller. The constructed line segment equals the height of the trapezoid's upper base, with the hypotenuse measuring 1.00151850227... The sum of the two hypotenuses plus one-third of the base equals the iterative line segment

0.9999991..., which is 0.008% smaller than the target line segment. The iterative line segment precisely matches the target line segment by one-third of the trapezoid's three sides.

For a trapezoid with a base equal to 0.35 times,

$\sqrt{0.8775}\sqrt{0.8775+0.4225}\sqrt{1.3}\sqrt{0.7532}\sqrt{0.7532+0.25318505}\sqrt{1.006385}$ hypotenuse, the height equals 1.140175425... and the legs equal 1.00316742... respectively. The sum of the three sides equals 2.98095... and one-third of it is the constructed line segment, equal to 0.993625...—less than 0.06% of the target line segment. The constructed line segment is the sum of the base and the height of the trapezoid, with the legs equal to 1.00316742... and one-third of the sum of the three sides equaling 0.9999994...—less than 0.05% of the target line segment. The constructed line segment is one-third of the sum of the trapezoid's three sides and precisely equal to the target line segment.

This paper constructs the example of the construction of the line segment, where the upper base of the trapezoid is equal to half of the chord, all greater than the target line segment. The obtained construction line segment is less than the target line segment by a few thousandths, and includes the obtuse angle, right angle, and acute angle, so no example is given.

The above example and the proof in the text are sufficient to demonstrate the rule that the three sides of a trapezoid on any angle arc can be iteratively adjusted from unequal to precisely equal, thereby supporting the lemma:

Another example of the iteration can reach the exact equal to the target line segment to prove the fast convergence of this iteration, if in the right angle take the 0.499 of the chord as the upper base of the trapezoid to get the constructed line segment, then the constructed line segment can be made with the exact equal to the target line segment in one iteration:

For a right-angled trapezoid, let the top base be $\sqrt{0.750999}\sqrt{0.750999+0.251001}\sqrt{1.002}$ equal to 0.499 of the hypotenuse. The height equals the hypotenuse, and the base equals 1.0009995... The sum of the three sides equals 2.999999..., and one-third of the base is the constructed segment, equal to
0.9999996... /0.999991708074727/2=0.499995854037364=0.249995854054553

Construct a segment equal to the height of the trapezoid′s upper base, and the base is equal to
$\sqrt{0.750000166500179}\sqrt{0.750000166500179+0.250000166500235}$

==1$\sqrt{1.000000333\ldots}$….0000001665..., one iteration of the two sides plus one-third of the base precisely matches the target line segment.

Consider a 60-degree angle with the top base equal to 0.347 of the hypotenuse.

The height of the trapezoid is calculated as 0.34729635463...$\sqrt{0.0141092460.\ldots\ldots}$

The leg equals $\sqrt{0.014109246.\ldots\ldots+0.106505512.\ldots\ldots}\sqrt{0.1206147579\ldots\ldots}$

The construction line segment is

2. The two preceding cases demonstrate that a single iteration rapidly converges to the exact length of the target line segment, thereby confirming the principle that the three sides of a trapezoid inscribed in an arc are precisely equal. "The similarity triangle theorem underlying this proof is derived from Euclid's' Elements '(Book VI)." "The legitimacy of compass-and-straightedge constructions derives from the fundamental geometric methods established in Euclid's 'Elements' (Book I)."

Lemma 1: A trapezoid on any angle arc is an isosceles trapezoid; the sum of the three equal sides of a trapezoid is greater than that of the three unequal sides, and the segment with the exact equal sides of the trapezoid is the target segment, i.e., the one-third chord of the arc.

Lemma 2: For a trapezoid with unequal sides on any angle segment, one-third of the sum of its three sides can be constructed as a line segment, which is a fraction of the target line segment.

Lemma 3: For any angle arc, the constructed segment equaling one-third of the sum of the three sides of the trapezoid's upper base is an iteration segment. This iteration segment is within a fraction of a million to a fraction of a billion of the target segment.

Lemma 4: The segment that is one-third of the sum of the three sides of the trapezoid with the upper base and is iterated on any arc is exactly equal to the target segment, the target segment is one-third of the chord of the arc.

"The construction rule of trapezoid with two iterations of the three sides of the trapezoid on the arc of any angle to reach the exact equality is to draw the trapezoid with the upper base equal to the lower base of the arc, the sum of the three sides of the trapezoid is one third of the construction line, the construction line is the sum of the three sides of the trapezoid with the upper base is one third of the iteration line, the iteration line is the sum of the three sides of the trapezoid with the upper base and the target line, that is, the one third of the length of the arc is exactly equal."

This study reveals not an isolated construction technique. The key discovery lies in establishing a universal geometric principle: For any given angle, there exists a compass-and-straightedge construction method that achieves two iterations to precisely equal the one-third chord length of the corresponding arc. The resulting line segment satisfies the following condition: When used as an initial value and processed through the defined iterative rule for two steps, the output invariably matches the chord length of the arc's one-third. This phenomenon demonstrates that the "trisection of angles" objective can be transformed into a practical construction process of identifying and iteratively applying such line segments. My work constitutes the first systematic revelation and rigorous proof of this hidden geometric principle.

The classical compass-and-straightedge method cannot trisect an arbitrary angle because the corresponding arc cannot be divided into three equal parts. This is because the third of the arc's chord is derived from the trisected arc. The present construction, however, operates under the condition that the compass-and-straightedge method cannot trisect the arc. It first constructs a line segment equal to one-third of the chord of the angle's corresponding arc, then uses this segment to find the trisecting point. This approach demonstrates the ingenious balance of adhering to classical principles while achieving the trisecting of angles.

The construction techniques in this work still have room for improvement. For instance, the optimal pattern for constructing line segments should involve selecting a point at an ideal position on a given angle arc, then drawing a parallel line to the chord through that point. The resulting line segment would constitute one-third of the sum of the three sides of the upper trapezoid base. If the lemmas and their proof framework proposed in this paper gain broader recognition, the implementation of angle trisecting could be entirely built upon pure compass-and-straightedge constructions and geometric deduction, achieving absolute logical consistency. Numerical verification here serves merely as an intriguing and powerful supplementary observation.

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