Journal of Applied Mathematics and Computation, 2025, 9(4), 266-271
https://www.hillpublisher.com/journals/JAMC/
ISSN Online: 2576-0653
ISSN Print: 2576-0645

Usage Frequency on Historic Stone Stairs: A Quantitative Model Based on Archard's Wear Law

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How to cite this paper: Ruiying Cao, Hongyan Zheng, Jiahang Wu, Wenchao Zhang. (2025) Usage Frequency on Historic Stone Stairs: A Quantitative Model Based on Archard's Wear Law. *Journal of Applied Mathematics and Computation*, 9(4), 266-271.
DOI: 10.26855/jamc.2025.12.006

Received: October 31, 2025
Accepted: November 30, 2025
Published: December 31, 2025

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Abstract

This study develops a novel quantitative model to estimate the historical usage frequency of stone stairs by integrating Archard's wear law with dynamic systems theory. The model establishes a mathematical relationship between measured wear depth and cumulative usage frequency, incorporating key parameters such as material wear coefficient, error correction factors, and standardized biomechanical load. This approach provides a rapid, low-cost computational alternative to traditional archaeological wear analysis techniques. Numerical simulations in MATLAB validate the model's ability to accurately reproduce characteristic wear patterns, including central tread wear and differential wear between ascent and descent paths. A sensitivity analysis identifies and ranks the influence of critical input parameters on the outputs. The paper concludes with a discussion of the model's limitations, primarily regarding parameter estimation in archaeological contexts, and suggests future research directions such as integration with 3D scanning data. This work contributes to digital heritage conservation by offering a tool for inferring historical pedestrian traffic.

Keywords

Archard's wear law; Dynamic systems; Usage frequency; Visual simulation; Interdisciplinary research

1. Introduction

Wear traces on historic building stairs provide critical physical evidence for reconstructing spatiotemporal patterns of ancient human activity. As transitional elements connecting functional spaces, stairs record cumulative crowd flow and behavioral habits through surface material degradation.

In current archaeological research, the analysis of stair usage frequency often relies on the comprehensive application of multiple techniques. For instance, Roux established a dynamic systems framework to interpret the evolution of technological usage behaviours [1]. In the field of stone heritage wear studies, a mature methodological system has been established, involving the acquisition of morphological data via high-precision 3D scanning combined with mechanical models for inverse analysis [2]. Additionally, radiocarbon dating, micro-wear analysis, and GIS spatial modeling are widely used to construct temporal sequences and spatial patterns of stair use [3]. Although these methods offer high precision, their strong dependence on specialized equipment and complex data processing workflows limits rapid application in field archaeology.

However, these existing methods are limited by their high reliance on specialized equipment and complex data processing workflows, which hinder rapid application in field archaeology. By integrating theories and methods from physics, mathematics, and archaeology, this study proposes a quantitative mathematical model to estimate stair usage frequency based on wear depth. By integrating Archard's wear law from physics with dynamic systems, we derive an explicit relationship between material degradation and cumulative usage cycles. This model aims to provide archaeologists with an efficient, low-cost tool for preliminary site assessment and to offer a new quantitative paradigm for interdisciplinary wear analysis. This original work was awarded the Finalist prize in the 2025 Interdisciplinary Contest in Modeling.

2. Theoretical Framework

This research integrates Archard's wear law with dynamic systems to construct a quantitative inversion model. Archard's law provides the physical mechanism for material degradation, while dynamic systems offer a mathematical framework to analyze the cumulative evolution of wear under coupled factors.

Archard's wear law is a classical theory in the field of material wear, which quantifies the relationship between wear volume and external conditions [4]. The standard governing equation is:

$$Q = k \cdot \frac{FL}{H} \tag{1}$$

Here, Q represents the wear volume, k is the dimensionless wear coefficient indicative of the material's propensity for wear, F is the normal load applied perpendicular to the contact surface, L denotes the total sliding distance, and H signifies the material hardness. This law clearly states the fundamental relationship: wear volume is directly proportional to the normal load and the sliding distance, and inversely proportional to the material hardness.

In the study of wear on historic building stairs, it is considered that the stair materials are typically homogeneous, and the impact of microscopic variations on macro-scale analysis is negligible. Through reasonable simplification, we convert wear volume into the more readily measurable wear depth. Furthermore, variables such as the temporal dimension and usage frequency are incorporated to extend the model to dynamic scenarios. This provides a theoretical foundation for quantifying and analyzing the evolution of stair wear, thereby supporting subsequent models.

Dynamic systems describe the evolution of a system's state over time. This theory transforms the complex interrelationships among wear depth, time, and influencing factors into computable state equations. Thereby converting qualitative research into quantitative analysis. During subsequent development of the wear model, the definitions of state variables, the formulation of evolution functions, and stability conditions will be further refined, thereby completing the modeling framework.

3. Model Construction

The cumulative wear depth h arises from repeated footstep-induced wear. Per Archard's law (Equation 1), wear volume per footstep is proportional to load F and inversely proportional to material hardness (embedded in c_k). We define total usage cycles over the service life as $N = f \cdot t$ (where f = usage frequency, t = service life).

The effective wear per cycle is not constant: a higher usage frequency f concentrates foot placement in high-traffic zones and may alter gait dynamics. According to Archard's law, wear amount relates to load and material hardness. Combining this with the linear dependence of step wear intensity I on material property (c_k), external load (F), and service time (t), we express I as [2]:

$$I = \lambda \cdot f \cdot t \cdot F \cdot c_\mathrm{k} \tag{2}$$

where λ integrates biomechanical and probabilistic effects. Therefore, the total wear depth follows:

$$h \propto I \cdot N = \lambda \cdot F \cdot c_\mathrm{k} \cdot f^2 \cdot t^2. \tag{3}$$

Introducing a composite error-correction factor a (which absorbs systematic errors such as λ), the model equation becomes:

$$h = a \cdot F \cdot c_\mathrm{k} \cdot f^2 \cdot t^2 \tag{4}$$

Inverting for usage frequency yields:

$$f = \sqrt{\frac{h}{a \cdot F \cdot c_k \cdot t^2}} \tag{5}$$

Here, c_k is the material wear coefficient (units: MPa^{-1}, to be specified mm^2/N), whose value is inversely related to material hardness [5]. Where f represents the usage frequency of the stairs (unit: times/year). Where t denotes the estimated service life of the stairs (based on stratigraphic dating results, with a precision controlled within ±5 years; for instance, t for a specific Ming Dynasty staircase is determined as 120 years).

A force analysis is performed on the staircase (as illustrated in Figure 1). The results indicate that the force exerted by the human body on the stairs is primarily composed of two components. These consist of a vertically downward normal force and a horizontally backward static friction force, which collectively form the composite force at the interface between the human body and the stair tread.

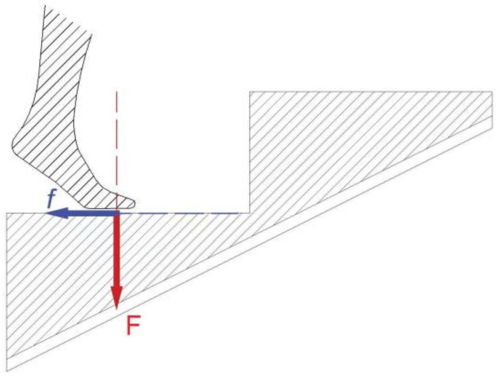


Figure 1. Force analysis on a stair step.

To simplify the model calculations and ensure the reasonableness of parameter values, we incorporate findings from ergonomics research on gait mechanics in adult populations, wherein the mean static normal force for an average adult is approximately 500-700 N [6]. During stair ascent and descent, additional loads are generated due to gait adjustments, and the magnitude of static friction typically accounts for 10%–15% of body weight. Therefore, this study estimates the resultant composite force exerted by the human body on the stairs to be 600 N [7]. This value ensures coverage of the load range for the mainstream population while avoiding excessive parameter dispersion caused by individual variations, thereby meeting the requirements for model simplification. Substituting 600 N for the normal load F perpendicular to the contact surface in Archard's law into Equation (5), the value of F directly influences the quantification of wear intensity m_k, thereby determining the accuracy of the estimated usage frequency and providing critical parameter support for subsequent calculations.

In line with dynamic systems theory, we define the model's state variable as wear depth $h(t)$ (evolving over time t). The evolution function is derived as

$$\frac{dh}{dt} = a \cdot F \cdot c_{\mathrm{k}} \cdot f^2 \cdot t \quad (6)$$

and stability is ensured by constraining parameters to physically meaningful ranges (e.g., $f > 0,\ c_{\mathrm{k}} > 0$), ensuring non-oscillatory, cumulative wear behavior consistent with archaeological observations.

4. Numerical Simulation and Pattern Analysis

For the aforementioned mathematical model, data visualization and simulation calculations were implemented in MATLAB, with the stair surface discretized at a grid resolution of 1 mm × 1 mm, and the wear depth was computed using an iterative algorithm with a step size of 100 usage cycles. The results are shown in Figure 2. As shown in Figure 2, the wear depth in the central region of the stair tread ranges from 0.8 to 1.2 mm, while that in the edge areas is only 0.2 to 0.5 mm. This quantitative disparity aligns directly with the gait characteristic that pedestrians tend to step predominantly on the central region of the stairs during ascent and descent. Combined with the site layout (e.g., the stairs connecting the sacrificial hall and the residential area), the high-frequency wear in the central region suggests possible fixed walking paths associated with daily transit or ritual activities, thereby providing a basis for inferring the functional use of the stairs [8].

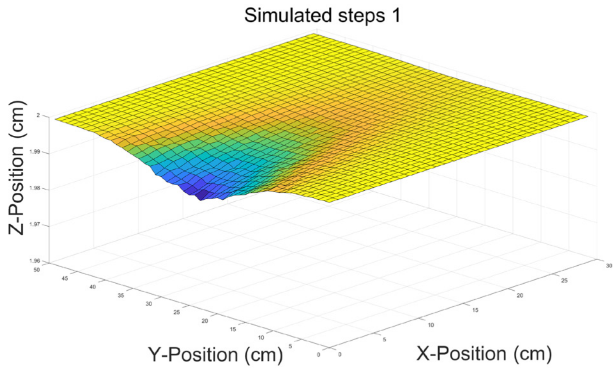
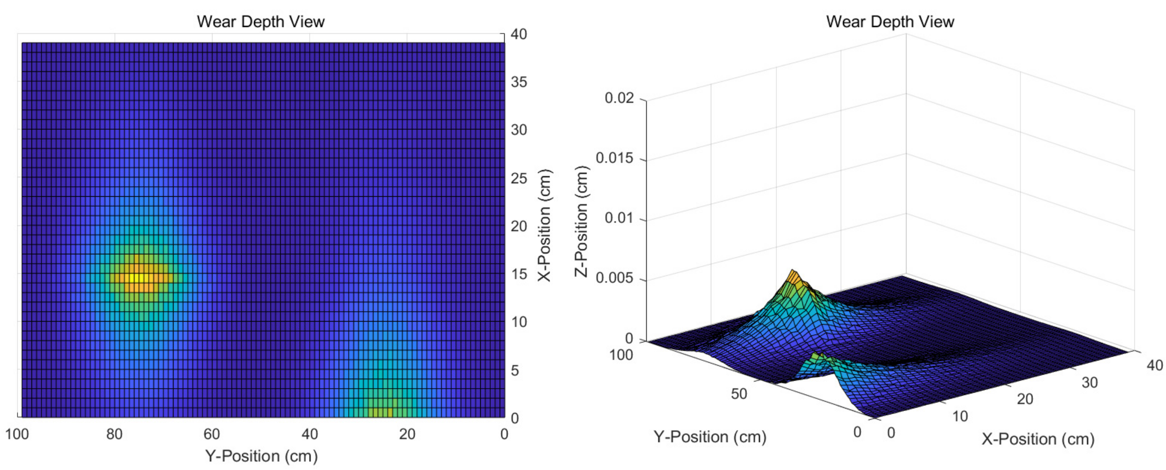


Figure 2. Stair wear simulation.

Building upon this foundation, the wear differential between ascending and descending usage scenarios is further investigated. During ascent, footsteps are predominantly concentrated on the outer edges of the treads, and due to the lower impact force associated with the gait propulsion mechanism, the resulting wear depth is comparatively smaller. During descent, footsteps are predominantly concentrated in the central region of the treads, and the greater impact forces exerted on the stairs result in significantly higher wear depth. Based on this quantitative pattern, the ratio of ascending to descending travelers can be inferred from the wear depth distribution across different zones of the stair treads. Simultaneously, an error term quantified by the correction coefficient a is incorporated into the model to account for practical uncertainties such as stochastic fluctuations in pedestrian foot placement within the archaeological context. When the measured wear depth contains a ±5% error, the incorporated error term confines the deviation of calculated results to within ±3%, thereby enhancing the model's practical utility. The corresponding simulation results are presented in Figure 3. Simulation results show greater wear depth in the descending area (left side) compared to the ascending area (right side) when stairs accommodate bidirectional flows. This pattern aligns with the gait biomechanical principle that descent generates higher impact forces on the stair treads.

Furthermore, simulation results for scenarios involving multiple individuals using the stairs simultaneously indicate that parallel wear depressions with comparable depths develop across the tread surface under such conditions. For two individuals walking side-by-side, two depressions form with a depth variation of ≤0.1 mm; for three individuals, three depressions appear with a depth variation also constrained to ≤0.1 mm [6]. Based on this pattern, the number and spacing of the principal wear depressions can be utilized to quantitatively infer the number of individuals using the stairs in parallel, thereby enriching our understanding of historical crowd activity patterns.



(a) Top view (Left: Ascending, Right: Descending) (b) Side view

Figure 3. Scenario with simultaneous ascending and descending traffic.

To evaluate the robustness of the proposed model under parameter variations and to identify key influencing parameters, a local sensitivity analysis is conducted. We focus on the model's core input parameters of usage frequency f (times/year) and service life t (years), observing their influence on the output variable, wear depth h(mm).

The benchmark values for the parameters are set as f=5 times/year and t=150. The parameters were then varied within a ±50% range (i.e., $f\epsilon[2.5,7.5]$, $t\epsilon[75,225]$, respectively) for a controlled variable simulation. The simulation results are shown in Figure 4.

The analysis results indicate that the model is highly sensitive to the frequency f, with wear depth h exhibiting a strong positive correlation (a non-linear monotonic increase) with f. When the frequency f increases from 2.5 to 7.5, the variation range of wear depth h exceeds ±1.0 mm. This finding underscores that in high-usage scenarios, an accurate estimation of f is crucial, as even slight measurement errors may lead to significant deviations in the inverted frequency. When the product of $f \cdot t$ is small (indicating low-frequency usage or short service periods), the absolute wear depth predicted by the model is also correspondingly small. In this region, even with certain parameter fluctuations, the resulting absolute error remains at a low level (<0.5 mm), demonstrating the strong robustness of the model. Conversely, when the product $f \cdot t$ is large, the model amplifies the effects of parameter variations, requiring users to provide highly accurate parameter inputs to ensure the reliability of the inversion results.

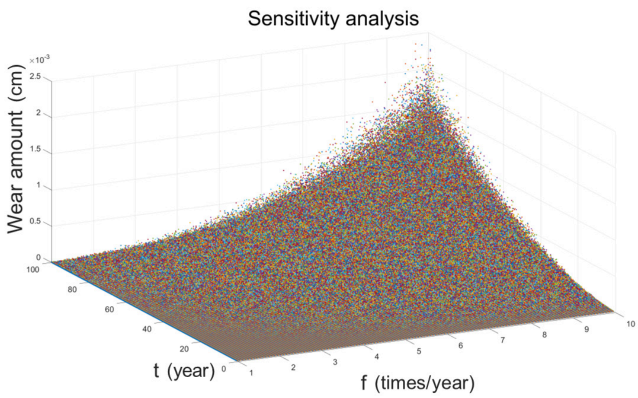


Figure 4. Sensitivity Analysis.

5. Conclusions and Future Work

This study establishes a simplified yet robust quantitative model for estimating the usage frequency of historic stone stairs by integrating Archard's wear law with dynamic systems. By introducing the material wear coefficient, error correction term, and standard human load, a mathematical relationship between wear depth and usage frequency has been established. Simulation results demonstrate that the model successfully reproduces key archaeological wear patterns. The most pronounced wear occurs in the mid-section of the stair treads, with significantly greater wear intensity observed during descent compared to ascent, while multiple parallel depressions form under multi-person traffic, corresponding to typical step widths, which validates the model's effectiveness for archaeological interpretation.

The primary limitations of the proposed model lie in its idealized assumptions, including the presumption of material homogeneity and the neglect of environmental influences. Future research may focus on three directions: first, incorporating environmental factors to develop a coupled wear-environment model; second, extending the model to account for non-homogeneous materials and composite structures; and third, integrating computer vision techniques to develop an automated image-based wear analysis system. These developments are expected to further enhance the model's applicability and practicality in archaeological practice, thereby providing more powerful analytical tools for wear archaeology studies.

Funding

This research was partially supported by the Guangdong Basic and Applied Basic Research Foundation (Grant No. 2022A1515110634) and the Teaching Quality and Educational Reform Engineering Project of Huizhou University (Project Nos. HuiYuanFa [2023] 158 and HuiYuanFa [2024] 173).

Acknowledgements

The authors wish to express their gratitude to the anonymous reviewers for their valuable suggestions, which significantly improved the quality of this paper.

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