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Predicting a Random Determinant with I.I.D. Negative Binomial Variates

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Abstract

This paper gives a probabilistic analysis of a determinant D of order 2 and 3 in which the elements are i.i.d. Negative Binomial variates. Using Chebyshev's inequality, fiducial limits (a type of statistical range used to estimate the uncertainty around a measured value) of D are obtained for order 2 and 3. The results may be compared with those obtained for other standard probability distributions. We are motivated to work in this problem considering the following applications: (1) Predicting the area of a triangle whose vertices are random variables. (2) In solving optimization problems where the coefficients in the objective function or the constraints are random variables. (3) Predicting the product of vectors (Cross Product) whose elements are random variables. (4) Predicting the equation of a plane containing two straight lines (in 3D) whose coefficients are random variables. (5) Predicting the Eigen values and Eigen vectors for random square matrices. (6) Predicting the solution of system of simultaneous linear equations, whose coefficients are random variables, using Cramer's rule.

Keywords

Random Determinant; Negative Binomial Distribution; Chebyshev's inequality
Mathematics Subject Classification: 62P99

1. Introduction

Negative Binomial distributions find applications in reliability and survival analysis. Some of the commonest examples of such behavior are the frequency distributions of plant density obtained by quadrant sampling when the clustering of plants makes the simple Poisson model inapplicable. It has been shown by different investigators that in such cases the Negative Binomial Distribution provides an excellent model because this distribution has a variance larger than mean. Yet another practical application of this model lies in Bacterial clustering (or contagion), e.g., deaths of insects, number of insect bites leads to the Negative Binomial Distribution, and the distribution also arises in inverse sampling from a binomial population or as a weighted average of Poisson Distribution. These applications motivated us to study a random determinant filled with i.i.d. (independently and identically distributed) Negative Binomial variates. We shall be using Chebyshev's inequality to predict such a random determinant. Chebyshev's inequality states that for a random variable X, $P\{E(X) - kSD(X) < X < E(X) + kSD(X)\} \geq 1 - \frac{1}{k^2}$, for some constant k which is usually some positive integer. Here E(X) is the mathematical expectation of X and SD(X) gives the standard deviation of X. The proof of Chebyshev's inequality can be found in any standard text on statistics (e.g Gupta and Kapoor [1]). See also Solary [2]. For a sound literature on determinants, see Muir [3].

2. Previous Work

Surprisingly there is not much work in this problem except for the paper by Wise and Hall [4] in which the distribution of a determinant of order 2 for i.i.d U (0,1) variates has been given. Saha and Chakraborty [5, 6] have extended the study for i.i.d U(0,θ) variates and i.i.d. U(1, 2, 3,….t) variates and, using Chebyshev's inequality, obtained the fiducial limits of D for order 2 and 3 for several other probability distributions. See also Agrawal, Pandey and Chakraborty [7].

3. Negative Binomial Distribution

Let f(x; r, p) denotes the probability that there are x failures preceding the r^{th} success in (x + r) trials. Now, the last trial must be a success, whose probability is p. In the remaining (x + r -1) trials we must have (r - 1) successes whose probability is given by binomial probability law by the expression: $\binom{x+r-1}{r-1} p^{r-1} q^{x}$.

Therefore, by compound probability theorem, f(x; r, p) is given by the product of these two probabilities,

i.e.
$$f(x; r, p) = \binom{x+r-1}{r-1} p^{r-1} q^{x} \quad p$$
$$= \binom{x+r-1}{r-1} p^{r} q^{x}$$

Hence, a random variable X is said to follow a Negative Binomial Distribution with parameters r and p if its probability mass function is given by:

$$P(X = x) = p(x) = \begin{cases} \binom{x+r-1}{r-1} p^{r} q^{x} ; x = 0,1,2,3 \dots \dots \dots \dots \\ 0 ; \text{otherwise} \end{cases}$$

For further literature on Negative Binomial and other probability distributions, see Gupta and Kapoor [1] and Gupta [8].

4. Our Contribution

First, consider a determinant D of order 2 as $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})$

Let a_{ij}'s be i.i.d Negative Binomial variates.

Remark: Mean of Negative Binomial Distribution $E(X) = \left(\frac{rq}{p}\right)$ (i)

and Variance of Negative Binomial Distribution $Var(X) = \left(\frac{rq}{p^2}\right)$ (ii)

For proof, see Gupta and Kapoor (2014).

Then,
$$E(D) = E(a_{11}a_{22} - a_{12}a_{21})$$
$$= E(a_{11}a_{22}) - E(a_{12}a_{21})$$
$$= E(a_{11})E(a_{22}) - E(a_{12})E(a_{21}) \quad \text{(Since, } a_{ij}\text{'s are independent)}$$
$$= \left(\frac{rq}{p}\right)\left(\frac{rq}{p}\right) - \left(\frac{rq}{p}\right)\left(\frac{rq}{p}\right)$$
$$= 0$$

Now,

We know that
$$Var(X) = E(X^2) - \{E(X)\}^2$$
$$E(X^2) = Var(X) + \{E(X)\}^2$$
$$= \left(\frac{rq}{p^2}\right) + \left(\frac{rq}{p}\right)^2 \quad \text{………………. using (i) and (ii)}$$
$$= \left(\frac{rq}{p^2}\right) + \frac{r^2q^2}{p^2}$$
$$= \frac{rq + r^2q^2}{p^2}$$
$$\therefore E(X^2) = \frac{rq(1 + rq)}{p^2} \quad \text{……………….(iii)}$$

Next

$\therefore$ Var(D)= Var $(a_{11}a_{22} - a_{12}a_{21})$

$= \text{E}(a_{11}a_{22} - a_{12}a_{21})^2 - \{\text{E}(a_{11}a_{22} - a_{12}a_{21})\}^2$

$\Rightarrow$ Var(D) = $\text{E}(a_{11}a_{22} - a_{12}a_{21})^2 - 0$

$\Rightarrow$ Var(D) = $\text{E}(a_{11}a_{22} - a_{12}a_{21})^2$

$\Rightarrow$ Var(D) = $\text{E}(a_{11}^2)\,\text{E}(a_{22}^2) + \text{E}(a_{12}^2)\,\text{E}(a_{21}^2) - 2\text{E}(a_{11})\,\text{E}(a_{22})\,\text{E}(a_{12})\,\text{E}(a_{21})$

$\because$ (a_{ij}'s are independent)

$\therefore$ Var(D) = $\text{E}(\text{X}^2)\,\text{E}(\text{X}^2) + \text{E}(\text{X}^2)\,\text{E}(\text{X}^2) - 2\,\text{E}(\text{X})\,\text{E}(\text{X})\,\text{E}(\text{X})\,\text{E}(\text{X})$

$= 2\ \{\text{E}(\text{X}^2)\}^2 - 2\,\{\text{E}(\text{X})\}^4$

$= 2\left(\frac{\text{rq}(1+\text{rq})}{\text{p}^2}\right)^2 - 2\left(\frac{\text{rq}}{\text{p}}\right)^4$ ………………using(i) and (iii)

$= \frac{2\text{r}^2\text{q}^2(1+\text{rq})^2}{\text{p}^4} - \frac{2\text{r}^4\text{q}^4}{\text{p}^4}$

$= \frac{2\text{r}^2\text{q}^2((1+\text{rq})^2 - \text{r}^2\text{q}^2)}{\text{p}^4}$

$= \frac{2\text{r}^2\text{q}^2(1+\text{r}^2\text{q}^2+2\text{rq} - \text{r}^2\text{q}^2)}{\text{p}^4}$

$= \frac{2\text{r}^2\text{q}^2(1+2\text{rq})}{\text{p}^4}$

$\therefore$ Standard Deviation (D)

$$\sigma = \left\{\frac{2\text{r}^2\text{q}^2(1+2\text{rq})}{\text{p}^4}\right\}^{\frac{1}{2}}$$

$$\Rightarrow \quad \sigma = \frac{\sqrt{2}\,\text{rq}\,(1+2\text{rq})^{\frac{1}{2}}}{\text{p}^2}$$

$\therefore$ Using Chebyshev's inequality yields

$$\text{P}(\text{E(D)} - \text{k SD(D)} < \text{D} < \text{E(D)} + \text{k SD(D)}) \geq \left(1 - \frac{1}{\text{k}^2}\right)$$

$$\Rightarrow \quad \text{P}(-\text{k}\,\sigma < \text{D} < \text{k}\,\sigma) \geq \left(1 - \frac{1}{\text{k}^2}\right)$$

where $\sigma = \frac{\sqrt{2}\,\text{rq}\,(1+2\text{rq})^{\frac{1}{2}}}{\text{p}^2}$

Next, assume that D is of order 3.

Consider $\text{D} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

= a(ei–fh) – b(di–fg) + c(dh–ge)

= aei – afh – bdi + bfg + cdh – cge

where a, b, c………..i are i.i.d. Negative Binomial variates.

Proceeding similarly as in the above case when D was of order 2, we get

E(D) = E(aei – afh – bdi + bfg + cdh – cge)

= E(aei) – E(afh) – E(bdi) + E(bfg) + E(cdh) – E(cge)

Let $\text{E}(\text{a}_{\text{ij}}) = \left(\frac{\text{rq}}{\text{p}}\right) = \lambda$.

Then

$\text{E(D)} = \lambda^3 - \lambda^3 - \lambda^3 + \lambda^3 + \lambda^3 - \lambda^3$

$\Rightarrow$E(D) = 0 ………………..(iv)

So

$\text{Var(X)} = \text{E}(\text{X}^2) - \{\text{E(X)}\}^2$

$\therefore \text{Var(D)} = \text{E}(\text{D}^2) - \{\text{E(D)}\}^2$

$= \text{E}\{(\text{aei} - \text{afh} - \text{bdi} + \text{bfg} + \text{cdh} - \text{cge})^2\} - 0$ …………………using (iv)

$= \text{E}(\text{aei} - \text{afh} - \text{bdi} + \text{bfg} + \text{cdh} - \text{cge})^2$

$= \text{E}[(\text{aei} - \text{afh} - \text{bdi} + \text{bfg} + \text{cdh} - \text{cge})(\text{aei} - \text{afh} - \text{bdi} + \text{bfg} + \text{cdh} - \text{cge})]$

$$= E\,[a^2e^2i^2 - a^2efhi - abdei^2 + abefgi + acdehi - ace^2gi - a^2efhi + a^2f^2h^2 + abdfhi - abf^2gh - acdfh^2 + acefgh - abdei^2 + abdfhi + b^2d^2i^2 - b^2dfgi - bcd^2hi + bcdegi + abefgi - abf^2gh - b^2dfgi + b^2f^2g^2 + bcdfgh - bcefg^2 + acdehi - acdfh^2 - bcd^2hi + bcdfgh + c^2d^2h^2 - c^2degh - ace^2gi + acefgh + bcdegi - bcefg^2 - c^2degh + c^2e^2g^2]$$

$$=\left(E(X^2)\right)^3 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X)\right)^6 + \left(E(X)\right)^6 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X^2)\right)^3 + \left(E(X)\right)^6 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X)\right)^6 - E(X^2)\left(E(X)\right)^4 + \left(E(X)\right)^6 + \left(E(X^2)\right)^3 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X)\right)^6 + \left(E(X)\right)^6 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X^2)\right)^3 + \left(E(X)\right)^6 - E(X^2)\left(E(X)\right)^4 + \left(E(X)\right)^6 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X)\right)^6 + \left(E(X^2)\right)^3 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X)\right)^6 + \left(E(X)\right)^6 - E(X^2)\left(E(X)\right)^4 - E(X^2)\left(E(X)\right)^4 + \left(E(X^2)\right)^3$$

Let $\left(E(X)\right)^6 = A$

$$E(X^2)\left(E(X)\right)^4 = B$$

$$\left(E(X^2)\right)^3 = C$$

Then

$$\text{Var}(D) = C - B - B + A + A - B - B + C + A - B - B + A - B + A + C - B - B + A + A - B - B + C + A - B + A - B - B + A + C - B - B + A + A - B - B + C$$

$$\text{Var}(D) = 12A - 18B + 6C \qquad \text{…………………(v)}$$

$$\because A = \left(E(X)\right)^6 = \left(\frac{rq}{p}\right)^6$$

$$B = E(X^2)\left(E(X)\right)^4 = \frac{rq(1+rq)}{p^2}\left(\frac{rq}{p}\right)^4$$

and

$$C = \left(E(X^2)\right)^3 = \left(\frac{rq(1+rq)}{p^2}\right)^3$$

From equation (v)

$$\text{Var}(D) = 12A - 18B + 6C$$

$$\text{Var}(D) = 12\left(E(X)\right)^6 - 18\,E(X^2)\left(E(X)\right)^4 + 6\left(E(X^2)\right)^3$$

$$\therefore \text{Var}(D) = 12\left(\frac{rq}{p}\right)^6 - 18\,\frac{rq(1+rq)}{p^2}\left(\frac{rq}{p}\right)^4 + 6\left(\frac{rq(1+rq)}{p^2}\right)^3$$

$$= 12\,\frac{r^6q^6}{p^6} - 18\,\frac{r^5q^5(1+rq)}{p^6} + 6\,\frac{r^3q^3(1+rq)^3}{p^6}$$

$$= \frac{6\,r^3q^3}{p^6}\{2r^3q^3 - 3r^2q^2(1+rq) + (1+rq)^3\}$$

$$= \frac{6\,r^3q^3(1+3rq)}{p^6}$$

$$\text{Var}(D) = \frac{6\,r^3q^3(1+3rq)}{p^6}$$

∴ Standard Deviation(D)

$$\sigma = \left\{\frac{6\,r^3q^3(1+3rq)}{p^6}\right\}^{\frac{1}{2}}$$

∴ Using Chebyshev's inequality yields

$$P(E(D) - k\,SD(D) < D < E(D) + k\,SD(D)) \ge \left(1 - \frac{1}{k^2}\right)$$

$$\Rightarrow \quad P(-k\,\sigma < D < k\,\sigma) \ge \left(1 - \frac{1}{k^2}\right)$$

Where

$$\sigma = \left\{\frac{6\,r^3q^3(1+3rq)}{p^6}\right\}^{\frac{1}{2}}$$

5. Discussion

It is of interest to compare the results on fiducial limits of D obtained for Negative Binomial Distribution as inputs with those obtained for other probability distributions. We refer the reader to tables 1 and 2 (Saha and Chakraborty [6]).

Table 1. Fiducial limts of D with different distributions constituting its elements when D is of order 2

Elements of D (i.i.d.)	Fiducial limts of D (obtained using Chebyshev's inequality)
Discrete Uniform U(1, 2,..., k) variates	$P\left(-r\sqrt{\frac{[(k+1)(2k+1)]^2}{18}-\frac{(k+1)^4}{8}} < D < r\sqrt{\frac{[(k+1)(2k+1)]^2}{18}-\frac{(k+1)^4}{8}}\right) \geq 1 - \frac{1}{r^2}$
Continuous Uniform U[0, θ] variates	$P\left(-r\sqrt{\frac{7}{72}}\theta^2 < D < r\sqrt{\frac{7}{72}}\theta^2\right) \geq 1 - \frac{1}{r^2}$
Exponential variates (mean 1/θ)	$P\left(-r\frac{\sqrt{6}}{\theta^2} < D < r\frac{\sqrt{6}}{\theta^2}\right) \geq 1 - \frac{1}{r^2}$
Normal variates (mean μ; variance σ^2)	$P\left(-r\sigma\sqrt{2(\sigma^2+2\mu^2)} < D < r\sigma\sqrt{2(\sigma^2+2\mu^2)}\right) \geq 1 - \frac{1}{r^2}$
Binomial (n, p)	$P\left(-r\,np\sqrt{2(1-p)(2np-p+1)} < D < r\,np\sqrt{2(1-p)(2np-p+1)}\right) \geq 1 - \frac{1}{r^2}$
Poisson (λ)	$P\left(-r\,\lambda\sqrt{2(1+2\lambda)} < D < r\,\lambda\sqrt{2(1+2\lambda)}\right) \geq 1 - \frac{1}{r^2}$

Table 2. Fiducial limts of D with different distributions constituting its elements when D is of order 3

Elements of D	Fiducial limts of D
Discrete Uniform U(1, 2,..., k) variates	$P\left(-r\sqrt{6\left[\frac{(k+1)(2k+1)}{6}\right]^3-18\left(\frac{k+1}{2}\right)^4\left[\frac{(k+1)(2k+1)}{6}\right]+12\left(\frac{k+1}{2}\right)^6} < D < r\sqrt{6\left[\frac{(k+1)(2k+1)}{6}\right]^3-18\left(\frac{k+1}{2}\right)^4\left[\frac{(k+1)(2k+1)}{6}\right]+12\left(\frac{k+1}{2}\right)^6}\right) \geq 1 - \frac{1}{r^2}$
Continuous Uniform U[0, θ] variates	$P\left(-r\frac{\sqrt{5}}{12}\theta^3 < D < r\frac{\sqrt{5}}{12}\theta^3\right) \geq 1 - \frac{1}{r^2}$
Exponential variates (mean 1/θ)	$P\left(-r\frac{2\sqrt{6}}{\theta^3} < D < r\frac{2\sqrt{6}}{\theta^3}\right) \geq 1 - \frac{1}{r^2}$
Normal variates (mean μ; variance σ^2)	$P\left(-r\sqrt{6(\sigma^2+\mu^2)^3-18\mu^4(\sigma^2+\mu^2)+12\mu^6} < D < r\sqrt{6(\sigma^2+\mu^2)^3-18\mu^4(\sigma^2+\mu^2)+12\mu^6}\right) \geq 1 - \frac{1}{r^2}$
Binomial (n,p) Variates	$P\left(-r\sqrt{6n^3p^3(1-p)^2(3np-p+1)} < D < r\sqrt{6n^3p^3(1-p)^2(3np-p+1)}\right) \geq 1 - \frac{1}{r^2}$
Poisson (λ) variates	$P\left(-r\sqrt{6\lambda^3(1+3\lambda)} < D < r\sqrt{6\lambda^3(1+3\lambda)}\right) \geq 1 - \frac{1}{r^2}$

6. Conclusion

1) If D is of order 2 with i.i.d Negative Binomial variates as input,

$$P(-k\,\sigma < D < k\,\sigma) \geq \left(1 - \frac{1}{k^2}\right)$$

where

$$\sigma = \frac{\sqrt{2}\,rq\,(1+2rq)^{\frac{1}{2}}}{p^2}$$

with

$$E(D) = 0 \text{ and } Var(D) = \frac{2r^2q^2(1+2rq)}{p^4}$$

2) If D is of order 3 with i.i.d Negative Binomial variates as input,

$$P(-k\,\sigma < D < k\,\sigma) \geq \left(1 - \frac{1}{k^2}\right)$$

where

$$\sigma = \left\{\frac{6\,r^3q^3(1+3rq)}{p^6}\right\}^{\frac{1}{2}}$$

with $$E(D) = 0 \text{ and } Var(D) = \frac{6\, r^3 q^3 (1 + 3rq)}{p^6}$$

Ethical Statement

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