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Identifying Cantor's Diagonal Argument as an Antinomy: Exploring Complementary Analysis Techniques

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Abstract

It has been noted that self-referential and ambiguous definition formulas are accompanied by complementary self-referential antinomy formulas, which give rise to contradictions. This made it possible to reexamine the ancient antinomies, Cantor's Diagonal Argument (*CDA*), and the method of nested intervals, which is the basis for evaluating the existence of uncountable sets. *CDA* is seen by many mathematicians as a beautiful and easy argument whose consequences lead to different powers of infinity, thus opening the door to mathematical paradise. The simple reasoning he uses seems to be a highly effective way of defining a sequence, implying a proof of uncountability because the diagonals of a two-character list turn into opposites, and at first glance, there is nothing to disprove the argument. A new look at the complementarity of Cantor's formulas in this article refreshes the hunches of Wittgenstein and other opponents of the existence of uncountable sets, putting *CDA* in a different light. In Cantor's theorem, a formula was used to define a set that cannot be the value of any argument of any function $f\colon \mathbb{N} \rightarrow P(\mathbb{N})$. Examining the complement of the created set, we find that this complement must be unique due to the bijective reversal $0 \leftrightarrow 1$ of the signs of the indicator function of the Cantor set. However, at the same time, its definition generates two different sets for one argument, which contradicts the basic property of every function. Other studies confirm the invalidity of Cantor's proofs and the nonexistence of uncountable sets.

Keywords

Cantor; diagonal; argument; method; uncountable sets; self-reference; paradox; antinomies; ambiguities; hypodox; dual; complementary; complements; a new hypothesis

1. Introduction

1.1 Paradox - antinomy

The meanings of both terms are similar and often appear interchangeably, although it is worth emphasizing the differences because they inevitably appear and their consequences can be different and significant.
Both terms signal emerging contradictions.

A paradox is a statement that contradicts our expectations and intuitions and can be both true and false.

An antinomy (ancient Greek: antí "against" + nómos "law") is a statement, a formula that generates contradictions. The contradiction can result directly from the formula or be deduced during logical deduction.

In any scientific discipline, the acceptance of a contradiction is untenable, as it precludes the discovery of objective truths. In such scenarios, every proposition would be equivalent to its antithesis. Within the realms of mathematics and other logical sciences, when faced with a paradox, it is imperative to elucidate its characteristics and exclude it from practical application and consideration if it results in a contradiction. On the other hand, antinomy as a source of contradiction should be excluded from the outset of all logical considerations, because the conclusions drawn may support both the thesis and the antithesis.

Alexander F. Shand in his article "The Antinomy of Thought" [1] from 1890 notes: "The antinomy does not belong to the affirmed reality but to our thought" and adds: "We shall have difficulty in abandoning both the thesis and the antithesis, which together constitute it; and any synthesis which would harmonize its contradictory propositions seems impossible."

Using Shand's idea of antinomy, its artificial and false origin, created in the mind, transmitted through artificial language, the purpose of which was initially to reflect reality, but this did not always happen - sometimes the mind plays tricks on us, and speech unconsciously creates events and objects unrelated to the real world. But sometimes we do it on purpose and not necessarily to distort reality. This does not have to be such a bad thing because it allows us to enjoy literature, music, or paintings, today even fantasy films.

Unlike antinomies, **a paradox** can be firmly rooted in reality, stand in opposition to our experience, and negate our understanding of surrounding events and facts, although it can also concern the products of our mind: thoughts and spoken sentences, which in a sense are also reality.

1.2 The meaning of antinomy in mathematics

Paradoxes are permissible in mathematics and the exact sciences, provided that they do not engender a contradiction within their logical framework. An example of such a paradox would be a picture consistent with the art of perspective, in which we see parallel railway tracks converging at one point on the horizon; that is, we could say that they have a common point at infinity (which, by the way, is not necessarily paradoxical).

Another example would be the

"*Paradox of increased perimeter*":

if we encircle the equator of a beach ball with a thread, and then introduce an additional meter into the perimeter and stretch it evenly over the ball's equator, then a cat will calmly pass through the gap, over the ball, and under the thread. Which species would possess the capability to traverse beneath a thread encircling the Earth, considering that the planet is modeled as a perfect sphere, the thread is inextensible, and exactly one additional meter of length is added? Intuitively, potential candidates might include:
(a) a cat,
(b) a mouse,
(c) a worm, or
(d) a bacterium.

This problem presents an intriguing challenge for mathematicians.

Of course, there are many paradoxes, but in the further part of the article we will deal only with antinomies, i.e. formulas, products of human speech, and imagination, which have no connection with the reality surrounding us and constitute a flawed image due to the imperfection of human thought - although they often create the appearance of correct, logical constructions. Since they generate contradictions, they should be eliminated from logic and mathematics, because otherwise they could be used to prove opposing theses.

1.3 Self-reference

Self-reference is, in most cases, responsible for the emergence of antinomies. Self-referential structures often lead to contradictions, undermining the validity of some basic mathematical proofs. For example, Zbigniew Tworak emphasized that self-reference is one of the most difficult problems in mathematical logic because it creates paradoxes and contradictions that are difficult to formally resolve within classical logic [2]. Of course, not all self-referential sentences create antinomies. An example of this is given in the following sentence:

"This sentence is written using the Latin alphabet".

This is a self-referential sentence, true and unambiguous.

The replacement of the phrase *"Latin alphabet"* with *"Greek alphabet"* will become an unambiguously false sentence, as will the negation of the above sentence.

In the case of antinomy sentences, determining their logical values is impossible:

> "This sentence is false".

An attempt to assign a truth value to the above sentence means from the content of the sentence that it is a false sentence, and assuming falsehood means that it is true. And what about the negation of such an antinomial sentence? We will analyze this in the following.

1.4 Hypodox, dual to paradox

Many researchers have studied the negation (DUAL) of antinomial (often called paradoxical) sentences. The term HYPODOX was introduced by Peter Eldridge-Smith [3] as the name of a formula that is in contraposition to PARADOX, and he studied such paradoxes and their duals as:

- Liar's Paradox:
Liar - dual = Truthfull (called Truth-Teller)

- Russell's Paradox
$V = \{X: X \notin X\}$ = Set of All Self-Non-Containing Sets -
- dual $= V' = \{X: X \in X\}$ = Set of All Self-Containing Sets.

He also noted that paradoxes are overdetermined, i.e., two or more terms in the definition of the subject are contradictory. At the same time, hypodoxes are underdetermined, i.e., there is a choice between objects that should be defined. The contradiction generated by the paradox is unacceptable in reality, but objects generated by the hypodox may be acceptable to many people who see nothing wrong with more than one object fulfilling an unambiguous definition.

Alexandre Billon reported very well the historical development of the study of hypodoxes in "Paradoxical Hypodoxes" [4], where he also devoted much attention to the study of the nature of these duals and to checking whether they are as paradoxical as the source paradoxes themselves.

In addition to the liar paradox, he also analyzed Richard's paradox. He stated that the source lies in language and in the definitions of objects, and therefore also the duals of paradoxes, i.e., hypodoxes, sometimes meet the definition of a paradox, as a phenomenon of contradiction between equally valid hypodox versions, which means that they should be treated in some sense as paradoxes and not as existentially correct versions of reality. Richard's paradox [5] was supposed to show that there is a list on which a definition of a real number generated according to the Cantor diagonal rule is considered correct, of course, among other real numbers from the same list.

The set theory explanation that this is just a paradox is, however, different from the one given by Billon and consists of the fact that it is impossible to correctly determine whether a text, a definition, correctly defines an object (a real number in this case), i.e. it is impossible to select a good list of definitions of only real numbers that would constitute the basis for generating, by the diagonal method, a new real number not included in that list.

1.5 Uncountable sets?

Georg Cantor introduced uncountable sets, different powers of infinity in set theory, in two ways:

1st: Nested Intervals Method [6]

2nd: A more familiar method is the diagonal method [7] of defining objects outside the list of objects selected from a set of objects with the same properties. Cantor describes how to create new objects using the diagonal[1] method, defining a string outside the list of infinite two-character strings, using characters from the diagonal list. Then this method was carried over to reals and sets, which is especially simple when the list is binary 0 – 1 and the reals are also represented in binary with the addition of a separator separating the integer parts. The indicator functions[2] for all subsets of the set of natural numbers $\mathbb{N}$ are identical to the elements of the set T and were used in Cantor's theorem with a similar logic of argument.

mathematicians and philosophers contemporary to Cantor, such as his promoter Kronecker, Poincare, Brouwer, and Wittgenstein. The intuitionist Brouwer commented [8]:

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[1] http://www.logicmuseum.com/cantor/diagarg.htm

[2] https://en.wikipedia.org/wiki/Indicator_function

[3] https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

> "Given that the Cantor number itself is also an element in M, it is suspicious whether the Cantor number is well defined."

Ludwig Wittgenstein considered a modified version of the *CDA* [9] (in fact, it was not a modified version - just an intermediate and necessary step in determining the Cantor number), showing its ambiguity and connections with self-referential paradoxes. These connections will be explored in detail using some of the most famous antinomies.

The diagonal method was also used to demonstrate Russell's antinomies. It led to the development of axioms that protected set theory from contradiction. To eliminate this antinomy from set theory, it was enough to eliminate only the universal set generated by the unrestricted understanding of predicates. However, there were reasons to look at self-referential complements (duals) of predicates, as we did for Russell's antinomy at the beginning of the article.

The axiom of specification[4] was considered the most important axiom for avoiding Russell's antinomy, and an important factor in preventing self-referential contradictions was the non-use of the set symbol (B) in the formula ϕ. However, this protection may be insufficient for formulas with implicit self-reference.

2. Known Antinomies from the Perspective of Their Duals in the Set Theory Approach

2.1 The Liar's Antinomy (Paradox)[5]

Let us divide people into two complementary sets:

A−liars and additionally: *A'−truthful ones*

Which set should include the person X who says:

X: "*I'm lying*" —-?

$X \notin A, X \notin A'$——— classical antinomy = contradiction.

In which set can we include the person Y saying: Y:

"*I'm telling the truth*" -?

$Y \in A'$- it is obvious, but if $Y \in A$, he would also have to say the same sentence:

"*I am telling the truth*", that is:

$Y \in A'$ *or* $Y \in A$,

which in turn gives ambiguity in the choice = we have an excess of implementation possibilities here.

2.2 Aunt's Antinomy – equivalent to the barber's paradox[6]

The paradox arises from the definition of

"*Aunt C only likes people who don't like themselves*"

Divide people into two complementary sets:

A − people who like each other and complementary:

A' − people who do not like themselves

$C \notin A, C \notin A'$ – antinomy = contradiction - no implementation possibilities.

Y − any other person (also *you*) can classify yourself according to your internal, current belief depending on, e.g., hairstyle, carcass, achievements, etc. to any set,

$Y \in A'$ *or* $Y \in A$ - excess of implementation possibilities = ambiguity of the definition

2.3 Russell's Antinomy[7]

In Russell's paradox, considering the property:

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[4] https://en.wikipedia.org/wiki/Axiom_schema_of_specification

[5] https://en.wikipedia.org/wiki/Liar paradox

[6] https://en.wikipedia.org/wiki/Barberparadox

[7] https://en.wikipedia.org/wiki/Russell

> “*Being your own element*”,

we can create two classes of sets: $V = \{X: X \notin X\}$, let us call it *normal class*, and a complementary class $V' = \{X: X \in X\}$, which we call *strange class* because it contains sets that contain themselves as elements. If class V were a set, is it strange or normal? If normal, then it should contain itself as a class of all normal sets, which, in turn, would contradict the fact that normal does not contain itself. The opposite assumption also leads to a contradiction, that is, we cannot qualify V for any of the classes. On the other hand, class V' may be normal if, of course, it was a set and then it would not contain itself because although it would collect only strange sets, it does not have to contain itself as an element.

The second possibility is that V' might be a strange class because it would contain itself as an element.

Self-reference is included in the definition of "strange set":

> »a set containing itself as an element«

and complementarily in the definition of "normal set":

> »a set that does not contain itself as an element«

These definitions would engender specific subsets within the ’set of all sets’ as characterized by naive set theory and can be identified utilizing the diagonal scheme, wherein all sets are arranged in a consistent sequence across both row and column headings. Subsequently, within the table at the intersections of the rows and columns, a value of 1 is recorded if the set represented by the column is an element of the set indicated by the row; conversely, a value of 0 is recorded if the set represented by the column is not an element of the set indicated by the row.

The indication on the diagonal of the list denotes whether the set is designated as normal, represented by 0, or strange, represented by 1. The set comprising all normal sets V would consequently be located in both row k and column k.

What symbol is appropriate for entry at the intersection a_{kk}? In this context, neither 0 nor 1 is allowed for insertion. **Contradiction = ANTINOMY**.

The collection of all peculiar sets V' would consequently appear in both the respective row m and the column m. In terms of the value to be inscribed at the intersection a_{mm}, both 0 and 1 are permissible entries. **Ambiguity**.

The above facts became the basis for the introduction of the new axiomatic, although considerations concerning the set V' were no longer necessary to introduce changes because the created axioms ZF and ZFC excluded the existence of the Universal Set (the Set of All Sets).

3. Hypothesis H

The above observations about the self-reference of formulas allow us to formulate a hypothesis:

> **H: Self-referential formulas that give rise to antinomian contradictions are accompanied by ambiguous self-referential supplementary formulas.**

The paradox disclosed by Russell in 1901 initiated the emergence of a system of axioms to prevent contradictions, putting an end to the naive set theory created by Cantor and his belief that simple predicates and formulas would be enough for defined objects to exist. Consider Cantor’s other ideas in light of the above hypothesis and the fact that the language in which they were articulated can describe more than just Platonic reality. The modern set theory includes not only the set theories based on the ZF or ZFC axioms[8] but also the earlier Cantor diagonal method in its various versions and generating various infinite powers along with the scale of alephs and betas.

It is true that Jules Richard, in 1905, that is, a few years after the publication of the diagonal method[9] by Cantor and a few years before the introduction of set-theoretic axioms, raised objections to this method, but an explanation for this paradox[10] was found[11].

To represent any rational number, all you need is a finite number of digits and a slash, optionally adding one minus sign, with the additional caveat that there is no zero under the slash. Irrational numbers, which extend rational numbers to real numbers, cannot be written in the same way.

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[8] https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory

[9] https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

[10] https://en.wikipedia.org/wiki/Richard%27s_paradox

[11] Error becomes an error when it is born as truth – Stanisław Jerzy Lec (Polish poet writer)

Although they often result from solving geometric problems of finite lengths, analogously to rational solutions, the Pythagoreans already noticed the difficulties in writing them precisely using digits only. They can be referred to by additional symbols such as π, but are often written as an infinite sum of fractions with a defined limit. Tarski[12] noticed that the language intended to describe and define objects, not only mathematical ones, based on a quite rich alphabet, does not always correctly define a given object, despite the appearance of truth. We will check the correctness of *CDA* later in the article.

4. *CDA* Cantor's Diagonal Argument [7]

We will consider Cantor's lemma[13] in modern notation where the characters m,w are replaced by the characters 0,1, and the character sequences E_1,E_2, ... are replaced by the symbols s_1, s_2, ..., and the set M of all sequences is the set T.
The antidiagonal E_0 is replaced by the symbol s.

Infinite sequences created from the available binary alphabet form a given set and are indicator functions of all subsets of $\mathbb{N}$. The set T exists because if we can choose any subset of $\mathbb{N}$, then we can create the set T from them. (The axiom of the existence of a set of all subsets).

It is not known whether this is a countable set, i.e., one that can be written in a list. Cantor creates a definition that he believes that, for any list, will create an antidiagonal sequence s, which is an element of T, and which sequence s cannot be in the list.

Since Cantor knows the result of his reasoning and believes that no one will provide him with such a complete list, it is redundant for him to assume that the list contains all the elements of the set T, and he does not do it.

To demonstrate the flaw in the diagonal method, it is enough to indicate a list for which Cantor's definition will not work.

In the lemma, Cantor suggested[14] that for any list taken from a set T that contains all binary, infinite strings, he could construct a binary string s from T not contained in that list. *CDA* seems to be an extremely simple way to define the string s since there are only 0s and 1s in the entire binary list, of the characters from the diagonal of the list are selected and converted from 0 to 1 and from 1 to 0.

Summarizing these remarks formally in symbolic notation, we will first specify the meaning of the terms used in *CDA*:

4.1 Terminology in CDA

alphabet = characters available for writing concepts;
binary alphabet = set of two characters 0,1
descriptive alphabet = a finite set of all possible signs for communication and writing
string = sequence = infinite sequence of characters from the binary alphabet
T = the set of all possible strings
s_i = the string symbol taken from T and placed in the *ith* line of the list.
list = array = strings arranged one below the other = $\mathrm{N} \ni i \rightarrow s_i \in T$
a_{ij} = symbol of the *jth* character of the *ith* string in the list
s_d = diagonal = the string that is on the diagonal of the list
s = antidiagonal = a string created by changing the signs of the diagonal
DEFs_i = descriptive string definition = a recipe using the descriptive alphabet for string construction, allowing you to specify all the characters of the defined string.
DEF s_d = diagonal definition = recipe on how to get characters from any list to generate a diagonal string
DEF s = antidiagonal definition = recipe on how to replace diagonal characters and generate antidiagonal characters

4.2 Definitions

Another rather trivial, but important note about strings and their definition:

Strings are identical if they have identical marks in the same positions; however, string definitions are equivalent if they specify identical strings, and it does not matter for the uniqueness of the strings which of the equivalent definitions we choose to define them.

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[12] https://en.wikipedia.org/wiki/Tarski%27s_undefinability_theorem
[13] http://www.logicmuseum.com/cantor/diagarg.htm
[14] https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument#cite_ref-10

set T = {infinite strings consisting only of zeros and ones}

List S of elements taken from T:

$s_1 = a_{11}a_{12}a_{13}a_{14}a_{15} \ldots$

$s_2 = a_{21}a_{22}a_{23}a_{24}a_{25} \ldots$

$s_3 = a_{31}a_{32}a_{33}a_{34}a_{35} \ldots$

. . .

where $a_{nm} \in \{0, 1\}$

DEF s_d (diagonal):

$$s_d = d_1d_2d_3d_4d_5\ldots \tag{1}$$

where $d_i = a_{ii}$

DEF s (antidiagonal):

$$s = a_1a_2a_3a_4a_5\ldots, \tag{2}$$

where $a_i = 1 - d_i = 1 - a_{ii}$[15]

At this point in the article, we must make a good-faith assumption:

ASSUMPTION: The definition of s is well-defined by Cantor's lemma

And this implies:

Antidiagonal s is well-defined by DEF s, when diagonal s_d is well defined by DEF s_d

Word definition of the sequence s:

DEF s:

The sign at the nth position of the sequence s differs by one from the *nth* sign of the *nth* sequence of the list being examined.

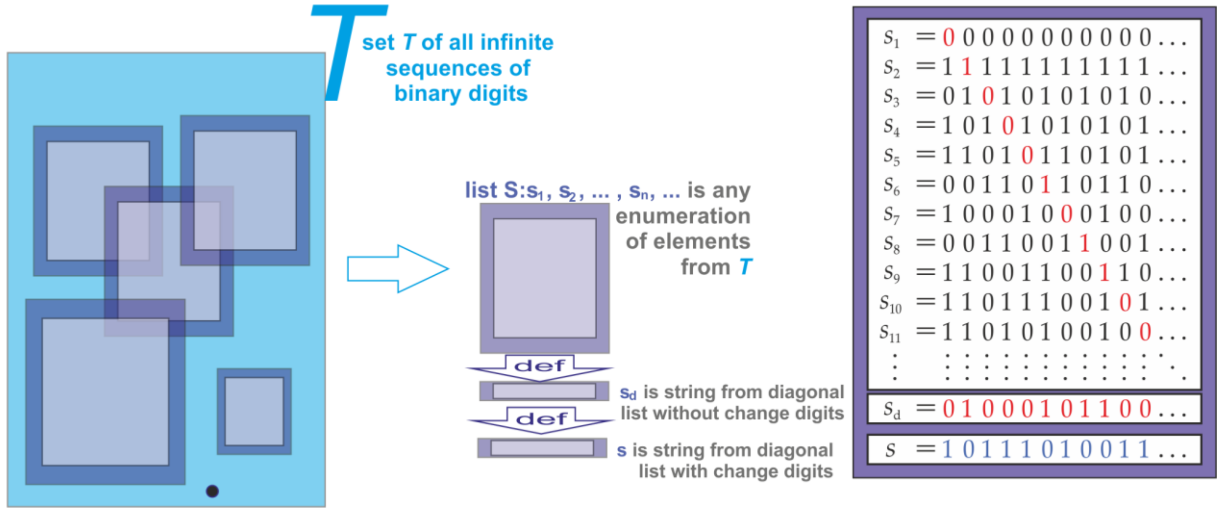
and his dual:

DEF s_d:

The sign at the nth position of the sequence s_d is identical to the sign of the *nth* sign of the *nth* sequence of the list examined.

4.3 Visualization of the *CDA* process for sequences

CDA = The process of generating sequences s for any selected list S of sequences coming from T is illustrated in the following figure, along with an example list:



$s_1 = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots$

$s_2 = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots$

$s_3 = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots$

$s_4 = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots$

$s_5 = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots$

$s_6 = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots$

$s_7 = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots$

$s_8 = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots$

$s_9 = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots$

$s_{10} = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots$

$s_{11} = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots$

$\vdots \quad \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots$

$s_d = 0\,1\,0\,0\,0\,1\,0\,1\,1\,0\,0\ldots$

$s = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1\ldots$

Figure 1. Process CDA and an example.

CDA consists of two successive steps:

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[15] The formula proposed by Wittgenstein. Here, of course, in this formula we will treat the digits 0 and 1 as numbers and perform operations on them, and the numerical result of subtraction will be converted back to a digit.

(1) Extracting characters on the diagonal of a list - creating an *sd* string.

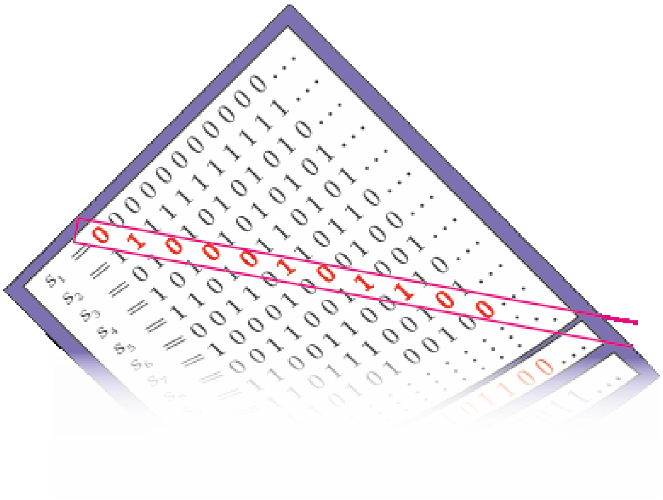

Figure 2. Extracting diagonal.

(2) Replace characters in the string s_d - Create the string *s*.

The first step is often unexposed and omitted, and one can even imagine that the characters on the diagonal are first swapped and then placed into the string *s* but the result must be the same regardless of the order of these operations that make up the Cantor string *s*. The existence of the string *s* is conditioned by the existence of the string s_d.

The isolation of characters on the diagonal of the list must first be noticed, and combining them into a chain *s* and the subsequent conversion into a chain *s* is completely independent of the remaining characters in the list *S* that are not on the diagonal of the list.

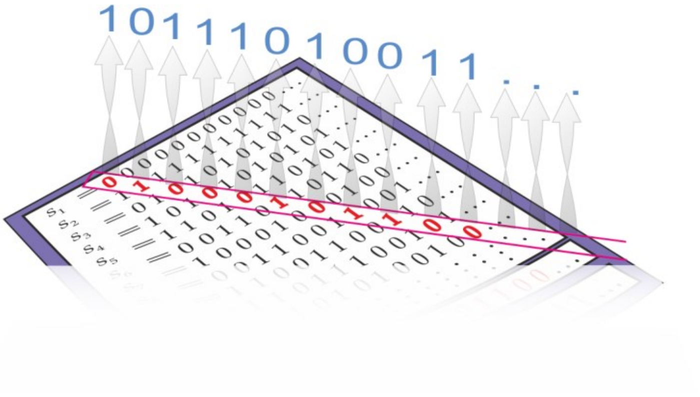

Figure 3. Changing digits from diagonal to antidiagonal.

It is worth noting: The uniqueness of the antidiagonal sequence is strictly dependent on the uniqueness of the diagonal sequence, since each character of the antidiagonal arises from a bijective conversion (0 ⟷ 1) of a character from the diagonal, and repeating the conversion from the antidiagonal gives the unique diagonal.

The above remark, made after assuming the correctness of Cantor’s definition, excludes the possibility that one of the paired strings: diagonal - antidiagonal has a unique interpretation, and the other has an ambiguous interpretation.

4.4 Equivalent definitions

It is important to distinguish the definition of a string, whose task is to generate string characters, from the string itself, which is a specific binary sequence. a): 00(0)...; b): 00(00)...; c) "empty set’s pointer function"; d) "complement of the sequence of only ones"; e) "the initial n characters form the initial sequence $<c_1c_2...c_n>$, and the next ones are the infinite sequence $<c_{n+1}c_{n+2}...>$, where $c_i = 0$ for each i" and as you can see, the same unique sequence can be defined with the help of an infinite number of definitions. This implies that a given definition may be substituted by another definition of equivalent nature.

The concepts of diagonal and antidiagonal are articulated as mathematical functions that map an input, represented by a list S, to their corresponding values, which consist of the sequences s_d and s derived from *T*. Given that all components of *T* can be systematically arranged into lists, it logically follows that the sequences ascertained through the definitions of diagonal and antidiagonal can similarly be incorporated into lists, with the possibility of appearing multiple times within certain segments of a randomly constructed list.

4.5 Assumption

ASSUMPTION:

> **We assume that Cantor's definition of antidiagonal always uniquely defines a sequence consisting only of allowed characters and contained in *T*.**

One might find this assumption unconventional, as it is evident that in a list composed only of zeros and ones, where there is a diagonal composed of the same elements, the antidiagonal formed through a bijective exchange 0 ←→ 1 could potentially be deemed invalid. However, this presumption enables Cantor to demonstrate that, notwithstanding the likelihood of inserting any sequence derived from *T* into any given row, such an endeavor is predisposed to failure.

This is attributable to the necessity for the sequence to possess a definition congruent with that of the antidiagonal, whereby it would manifest a discrepancy at each respective position within an infinite sequence precisely at the juncture within the sequence that aligns with its corresponding position in the list.

This means that the theoretical possibility of existence in a list of sequences defined by the antidiagonal definition or by an equivalent definition is admitted, and not, as is commonly believed, that the list should be constructed without the antidiagonal sequence because the *CDA* procedure is intended only for a ready and complete list without sequences defined by the antidiagonal definition.

Let us also note that the above assumption, although not articulated, is necessary, because if it were false, the conclusion about the existence of uncountable sets would also be false, despite the correct indirect observation about the absence of the antidiagonal sequence in the line of the list, in which only correct sequences can be placed.

We will see whether this assumption is correct later.

4.6 Dual sequence to *s* and its presence in a list

The set *T* consists of elements with infinite chains, so the diagonal must also have an infinite length, which means that the list *S* must also be infinite for the definition of the antidiagonal for a given list to have the same dimension as the elements of the set *T*.

Observing the various possibilities of defining the same sequence creates the possibility of repeating the presence of the same sequence on the constructed list (in any way) even more than once, which even requires checking in *CDA* whether the constructed list does not include a sequence defined differently.

Cantor's Diagonal Argument shows that an arbitrarily constructed list *S* of elements taken from *T* will never contain all the elements from *T*, because using the definition of *s* for a given list *S* it is always possible to reconstruct a string *s* that is not in this list. Of course, a simple list should also generate such a sequence *s*, and at the same time, it can contain the diagonal sequence s_d in some rows since it is not forbidden, and, in addition, it is possible even at several positions of the list.

For the simplest list, i.e. a list containing only zeros (a sequence of an infinite number of zeros in each row), a diagonal string will also be repeated at all positions in the list, also composed only of zeros.

Ludwig Wittgenstein[16] modified the definition of generating the string *s* and received the definition of the string s_d, but neither he nor Chaohui Zhuang [10], analyzing his reasoning [9], noticed that the definition of s_d is necessary, intermediate, and complementary to the definition of *s*. The modification was ignored because such a process may contain errors. Erdinç Sayan [11] also pointed out the necessity of the existence of the s_d string as a complement to the *s* string, revealing the paradoxical nature of their definition.

Creating new objects s_d, *s* described by *CDA* (strings, real numbers, or subsets) outside of a list requires building a proper list. In Cantor's lemma, the source of such a list of infinite two-character strings is simply the set *T* of all such strings.

4.7 Demonstrating the necessity of the existence of a list generating *s* and containing s_d

We have two cases of the presence of the string s_d in list *S*:

---

[16] Ludwig Wittgenstein, Edited by G. H. von Wright, Translated by G. E. M. Anscombe. Zettel. University of California Press, 1970.

(1) The string s_d defined by the characters on the diagonal is in the parsed list at some position k (green arrow).

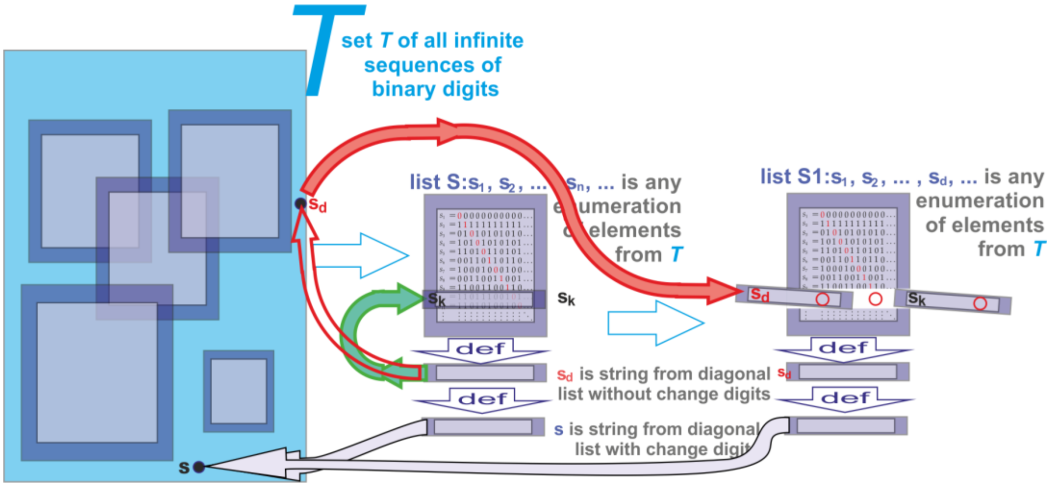

Figure 4. The existence of a list containing diagonal string for any antidiagonal.

(2) The string s_d is not in the list S (empty red arrow). This string is in the set T, from which we can take it and put it in place of any word in the list S, creating a new list $S1$. This substitution will not change the value of the signs on the diagonal, which means that the string s will not change either.

Note that, similarly to the case of other lines of the list, where the definitions determine the characters that appear in a given line of the list, the characters in the embedding line k should be determined by the diagonal definition s_d.

From the above reasoning, we conclude that for any sequence s, there is at least one list of sequences (S or $S1$) extracted from T, for which the definition s_d should uniquely define a two-character sequence in that list on some line of that list. The definition of this string s_d is as follows:

The sign at the nth position of the string s_d is identical to the sign at the nth position of the nth string of the examined list.

Symbolically, for a list of binary strings:

$$s_d = d_1 d_2 d_3 d_4 \ldots, \text{ where } d_i = a_{ii} \quad (\text{or: } a_{ii} = 0 \Rightarrow d_i = 0, \text{ and } a_{ii} = 1 \Rightarrow d_i = 1)$$

4.8 *CDA* for subset ℕ

Sequences consisting only of the sign 0 and sign 1, from the set T, are ready-made indicator functions[17] for the subsets of ℕ, the unambiguous presentation of which determines the correctness of the set definition.

The subset definitions of ℕ used in Cantor's Theorem[18] can be expressed in a symbolic language using the Axiom of Specification in the form: $\{x \in \mathbb{N}: \varphi(x)\}$. This axiom is supposed to guarantee the existence of the ℕ subset for all elements of ℕ satisfying the φ predicate, where the symbol of the defined set B does not occur freely in the formula φ.

This axiom was introduced into set theory to prevent Russell's paradox[19].

The Cantor definition $\{x \in \mathbb{N}: x \notin f(x)\}$ should define the set B for every function f: $\mathbb{N} \to P(\mathbb{N})$ outside the list f, and the supplementary definition $\{x \in \mathbb{N}: x \in f(x)\}$ should define the supplementary set $B' = \mathbb{N} \backslash B$.

Let us note:

Definition $\{x \in \mathbb{N}: x \notin f(x)\}$ uniquely defines the set B together with its indicator function $\mathbf{1}_B$ ⇔ definition $\{x \in \mathbb{N}: x \in f(x)\}$ uniquely defines the set $B' = \mathbb{N} \backslash B$ together with its indicator function $\mathbf{1}_{B'}$, having bijectively reversed signs $0 \longleftrightarrow 1$ to the indicator function $\mathbf{1}_{B'}$.

We shall scrutinize the initial segment of the collection of objects delineated by employing the previous chapter, we know that for each element from T, which is the result of the antidiagonal definition, there is a list that contains the diagonal of this list in at least one row. We will make two attempts to place the diagonal sequence defined by its definition in a fifth row.

Attempt I. We will construct a list f (defined subsets of ℕ):

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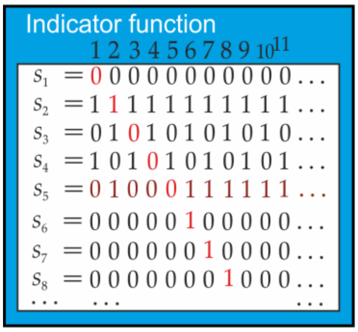
[17] https://en.wikipedia.org/wiki/Indicator_function

[18] https://en.wikipedia.org/wiki/Cantor's_theorem

[19] https://en.wikipedia.org/wiki/Russell%27s_paradox

$f(1) = \Phi \equiv s_1$
$f(2) = \mathbb{N} \equiv s_2$
$f(3) = \{x \in \mathbb{N} : x = 2m, m \in \mathbb{N}\} \equiv s_3$
$f(4) = \{x \in \mathbb{N} : x \in \mathbb{N} \setminus f(3)\} \equiv s_4$
$f(5) = \{x \in \mathbb{N} \setminus \{1, 3, 4, 5\}\} \equiv s_5$
for $n > 5 : f(n) = \{n\} \equiv s_n$

We show the indicator functions **1** of the defined subsets below for the above definitions.



Indicator function
1 2 3 4 5 6 7 8 9 10 11
s_1 = 0 0 0 0 0 0 0 0 0 0 0 . . .
s_2 = 1 1 1 1 1 1 1 1 1 1 1 . . .
s_3 = 0 1 0 1 0 1 0 1 0 1 0 . . .
s_4 = 1 0 1 0 1 0 1 0 1 0 1 . . .
s_5 = 0 1 0 0 0 1 1 1 1 1 1 . . .
s_6 = 0 0 0 0 0 1 0 0 0 0 0 . . .
s_7 = 0 0 0 0 0 0 1 0 0 0 0 . . .
s_8 = 0 0 0 0 0 0 0 1 0 0 0 . . .
. . . .

Figure 5. Indicator functions for formulas defined subset.

The characters on the diagonal of the list, highlighted in red, after being replaced with their opposites, will create an indicator function of set B, which cannot be on this list f and is defined by Cantor with the formula: $\{x \in \mathbb{N} : x \notin f(x)\} = B$.

Set B is uniquely determined for a given function f. It is permissible to replace the used set definitions with other definitions equivalent to them, i.e., those that do not change the elements of the defined set and thus do not change any sign of the indicator function and will not change any sign on the diagonal of the list and the generated antidiagonal. An example is the replacement of definition 1: replace the symbol of the empty set Φ with such a definition $\{x \in \mathbb{N} : x \in \mathbb{N} \setminus f(2)\} \equiv s_1$, noting that this set is the complement of the set defined in the next line. Equivalent definitions do not change any element in the table of pointer functions for this function f.

The relation between two complementary subsets is bijective 0 ↔ 1, and can also be seen for diagonal and anti-diagonal sequences. Let us now look at Cantor's antidiagonal definition for this list f: $\{x \in \mathbb{N} : x \notin f(x)\}$ generates an indicator function $\mathbf{1}_B$=101110(0)...

The set B', as a complementary set of $B' = \mathbb{N} \setminus B$, is defined by the complementary definition: $\{x \in \mathbb{N} : x \in f(x)\}$ with a diagonal indicator function $\mathbf{1}_{B'}$=010001(1)...

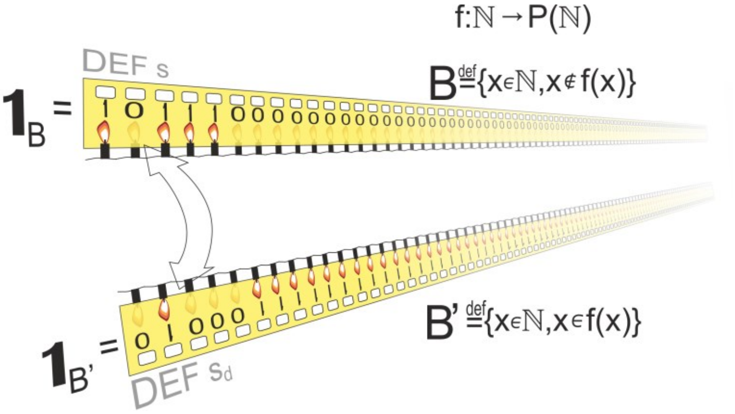


Figure 6. Illustration of the unambiguous dependence of mutually complementary sequences.

The bijective dependence of the diagonal and antidiagonal indicator functions can be imagined as an infinite string of Christmas tree lights placed along a film strip on either side of the strip, where the lit bulbs signal the belonging of the sequence number of the light to a given sequence.

Since the lights do not go out if they are on the other side of the strip, looking from the other side, we see a presentation of the complementary set.

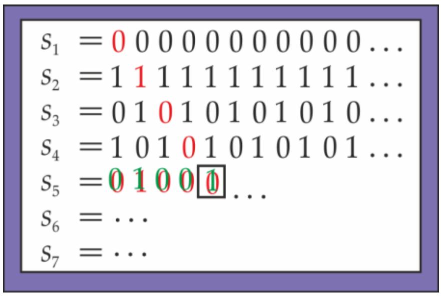
Having at our disposal a well-defined one of the above sequences, we can define the second one in yet another way by complementation:

$$B' \text{ well-defined} \Rightarrow B = \{x \in \mathbb{N} : x \in \mathbb{N} \setminus B'\}, \text{ and}$$
$$B \text{ well-defined} \Rightarrow B' = \{x \in \mathbb{N} : x \in \mathbb{N} \setminus B\}.$$

The diagonals of the indicator list are identical to the characters in the fifth row, which should mean that the general definition of the set B' is equivalent to the detailed definition of the fifth row and that the definitions can be changed:

$$f(5) = \{x \in \mathbb{N} \setminus \{1, 3, 4, 5\}\} = \{x \in \mathbb{N} : x \in f(x)\} = B'$$

Both of the above formulas should equally and uniquely generate a unique and bijective indicator function, depending on the uniquely determined antidiagonal indicator function.



$s_1 = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots$
$s_2 = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots$
$s_3 = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots$
$s_4 = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots$
$s_5 = 0\,1\,0\,0\,\boxed{0}\ldots$
$s_6 = \ldots$
$s_7 = \ldots$

Figure 7. Indicator functions for formulas defined by the Axiom of Specification.

Through a comprehensive analysis facilitated by pointer functions specifically developed for these formulas, it becomes evident that formula $\{x \in \mathbb{N}: x \in f(x)\}$, located on the fifth line, exhibits ambiguity. This ambiguity is manifested through two divergent character sequences (indicated by red and green), which are alternatively distinguished by the sign at the fifth position, aligning with the incorporation of the natural number "5" into the designated set B'.

Consequently, this formula fails to produce an unambiguous subset B'. Nevertheless, one of the sequences, namely 01000(1)..., perfectly aligns with the original definition of the sequence before its transformation under Cantor's diagonal definition. Regrettably, the alternative sequence, 01001(1)..., emerging from the diagonal definition's ambiguity, is unsuitable for serving as an example of a uniquely defined antidiagonal for our list. Initially, all characters on the list's diagonal were definitively characterized by their unique definitions, which inherently extended to the definition of all diagonal and antidiagonal characters.

However, it becomes apparent that replacing the prior context-independent definition of diagonal characters with this ostensibly unique definition results in a loss of uniqueness in determining the character situated at the convergence of the embedding line and the list diagonal. Hence, any permissible characters could be placed at this junction. This phenomenon is depicted in the accompanying figure (for real numbers from (0,1)), where gray arrows denote the transfer of well-defined characters from the diagonal to the embedding line k, leaving no definitive characters available for the empty square:

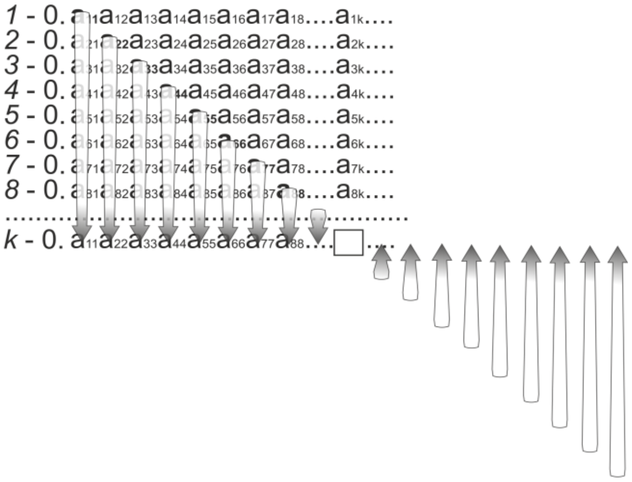


Figure 8. Real numbers. Inserting a diagonal definition in line k.

Assuming the correctness of the antidiagonal and diagonal definitions, by the conclusions of the previous chapter, that for each antidiagonal sequence, there is a list generating such an antidiagonal sequence that also contains a diagonal sequence in some line, we could try to create a list containing a diagonal sequence from the beginning, e.g. in line 5 of the list – this time based directly on Cantor's recipe. However, considering the reservations raised about Cantor's definition, and the need to introduce only well- and unambiguously defined sequences into the list, we must first check their correctness (by examining the indicator functions).

Attempt II. We will build a new list of f (defined subsets of $\mathbb{N}$):

$f(1) = \{x \in \mathbb{N} : x \in \mathbb{N} \setminus f(2)\} \equiv s_1$ ………………………………………………………**1**s_1 = 0000(0)…
$f(2) = \mathbb{N} \equiv s_2$ ……………………………………………………………………….. . . .**1**s_2 = 1111(1)…
$f(3) = \{x \in \mathbb{N} : x = 2m, m \in \mathbb{N}\} \equiv s_3$. ………………………………………………**1**s_3 = 0101(01)…
$f(4) = \{x \in \mathbb{N} : x \in \mathbb{N} \setminus f(3)\} \equiv s_4$ ……………………………………………………**1**s_4 = 1010(10)…
$f(5) = \{x \in \mathbb{N} : x \in f(x)\} \equiv s_5$. ………………………………………………………..**1**s_4 = 0100?…….
$for\ n > 5 : f(n) = \{n\} \equiv s_n$

Accompanying the aforementioned formulas, we present the corresponding indicator functions.

In substitution of ?, two distinct characters: 0 or 1, may be inserted. Consequently, this results in two distinct sets $s_5 = B'$, one of which includes the element '5' while the other does not.

Thus, these differing sets would represent the value of a single argument for the function f, contradicting the fundamental property of a function.

Upon initial examination, a sequence congruent with the diagonal sequence was inserted into the fifth line; however, it cannot be replaced by a universal diagonal definition derived from the antidiagonal definition, due to the introduction of ambiguity and its non-equivalence.

It is evident that the list consists solely of ones and zeros, as the definitions imposed upon it were unequivocal, ensuring that no "ambiguous definition" was present among them. The diagonal sequence, exemplified by 010001(1)..., is readily apparent yet not specified by the formula $\{x \in \mathbb{N} : x \in f(x)\}$.

Furthermore, one can readily compute the antidiagonal, as illustrated by the indicator function 10111(0)...

On the second attempt, it turns out that it is impossible to enter the sequence obtained thanks to Cantor's diagonal definition into the list of subsets because of its incorrect operation (IT ASSIGNS TWO VALUES TO ONE ARGUMENT).

Both attempts demonstrate that Cantor's diagonal argument is flawed due to issues of self-reference, specifically the absence of a clearly defined sign at the juncture where the sequence introduction line intersects with the list diagonal. We shall provide a detailed explication and verification of the functioning of the revised versions shortly. Before doing so, however, we will address the antidiagonal definition itself, as well as the implications of inserting the antidiagonal sequence into the list—an approach inspired by Wittgenstein's formula.

4.9 DEF *s* according to Wittgenstein's recipe

In the *CDA* lemma, Cantor, defining an antidiagonal sequence, uses the letters "m" and "w" as characters that create infinite sequences $E_1, E_2, \ldots E_i \ldots$ and that is in the set M of all possible sequences consisting of only these two letters. He then shows that for each list of elements taken from the source, he can find an element E_0 from M that is not on the list, and for this purpose, he uses a sequence extracted from the diagonal of this list with the signs changed to the opposite ones, which can be well illustrated by our example with an infinite film strip:

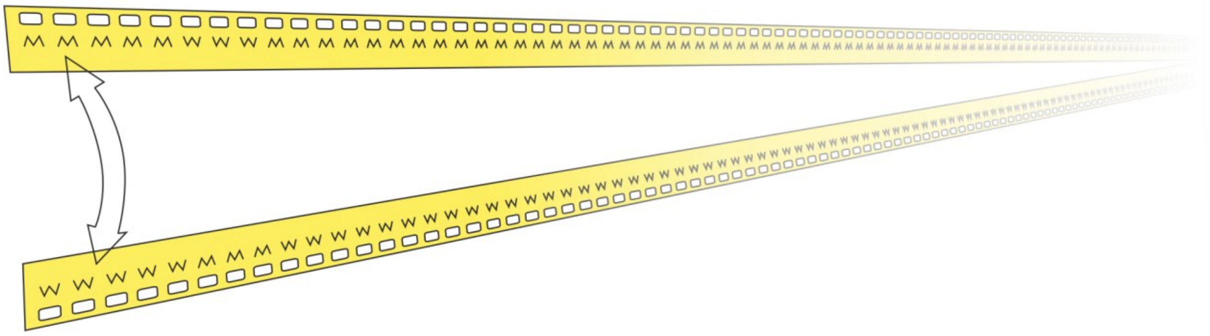


Figure 9. Cantor's complementary sequences.

Here we see how two complementary sequences remain in a bijective relation of $m \longleftrightarrow w$.

One can be the observed diagonal of the list, and the other antidiagonal - all you need to do is plot the signs of one of the sequences on a clear film strip and, observing the strip from the other side, see the complementary sequence.

The possibility of the existence of an antidiagonal sequence defined by Cantor's definition on the list resulted from the assumption and necessity of proving that for each list taken from the source set, even one that hypothetically contained in any row a sequence defined by Cantor's formula, this would be impossible due to the difference in signs at the position in the sequence that the sequence would occupy on the list.

The process of performing computations using these symbols within this framework encounters inherent difficulties. The contemporary approach of reformulating this lemma by employing the symbols "0, 1" in lieu of "m, w" facilitates a seamless transition to alternative variations of the diagonal argument. This methodology is also pertinent to the application of indicator functions for subsets of N. In addition, by prefixing with "0.", one can construct a binary set of real numbers confined within the interval (0,1).

Furthermore, the conversion of a diagonal sequence into an antidiagonal sequence can be explicated through the formula introduced by Wittgenstein:

$$s \equiv a_1 a_2 a_3 \ldots \text{, where } a_i = 1 - a_{ii}$$

a_{ii} is a character from the list of two-character sequences taken from the source set T (in Cantor's M) and located in the *i-th* row at the *i-th* position, i.e., on the diagonal of this list defined by the supplementary definition of diagonal:

$$s_d \equiv d_1 d_2 d_3 \ldots \text{, where } d_i = a_{ii}$$

In the preceding two chapters, it has been established that while a list should theoretically generate an antidiagonal sequence and encapsulate a diagonal sequence within a specific line of the list, defining it by the aforementioned definition of diagonal introduces ambiguity. This ambiguity, as informed by investigations into duals, hypodoxes, and hypothesis H, implies that the definition of antidiagonal might represent an antinomy. This supposition can be examined by attempting to integrate this definition among other definitions into an infinite list of sequences delineated by various definitions.

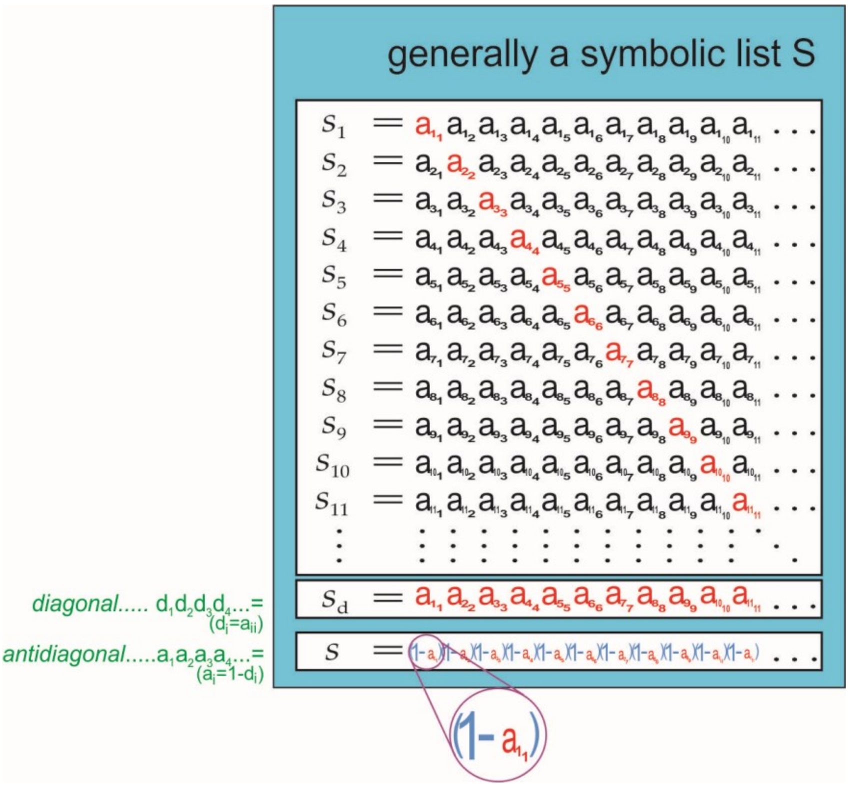


Figure 10. CDA in modern notation with Wittgenstein's formula.

Now, let us try to put the sequence resulting from the antidiagonal definition of s into line 7:

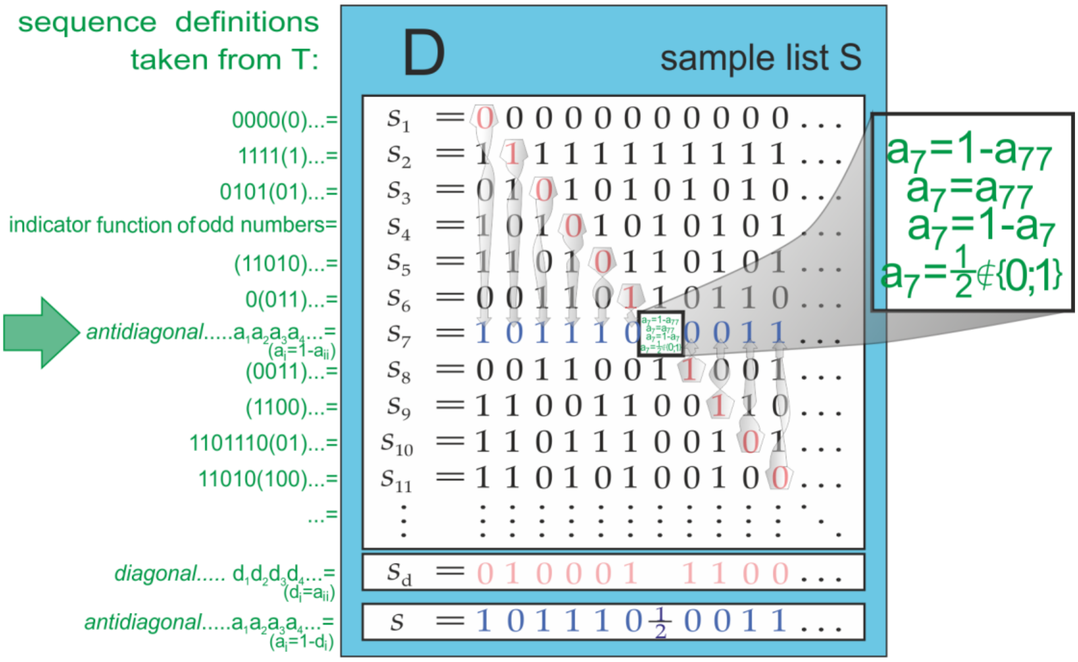


Figure 11. Solution for the antidiagonal formula.

It turns out that in this case, the definition of antidiagonal generates a ½ character in the 7th position, beyond the characters allowed in the list!

And we can only place sequences from the set *T* in the list, consisting only of zeros and ones!

$$\text{DEF } s \quad \Rightarrow s \notin T$$

This is not the definition of an element of the set *T*.

To put the diagonal sequence defined by the diagonal definition DEF s_d in a row of the list, we would have to take two sequences with different characters from the set T to the same place in the list (crossing the embedding row with the diagonal), which for the complementary antidiagonal sequence s meant that there were no characters available not selected by the diagonal definition, and trying to put in list the antidiagonal sequence defined by DEF *s* only confirms this, indicating that the character ½ is the only one possible to realize in the sequence while maintaining definitional correctness. However, this character is not allowed among the elements of *T*.

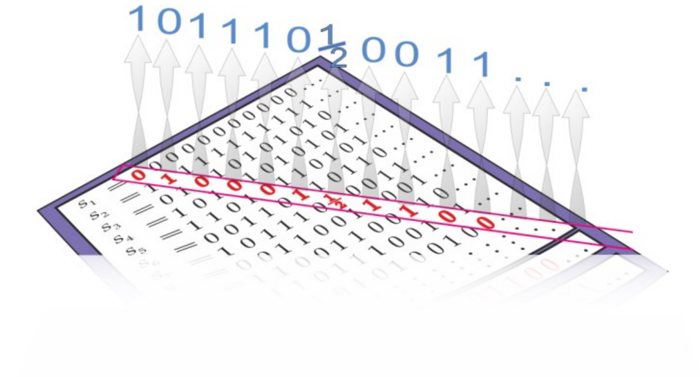


Figure 12. Sequence with a foreign character (not included in T).

If an antidiagonal sequence defined by DEF *s* can occur in a given list, finding an element of the set *T* is impossible because the formula does not generate a sequence contained in *T*.

Assuming that an antidiagonal sequence defined by DEF *s* can occur in a given list, finding an element from the set *T* is impossible because the formula does not generate a sequence contained in *T*.

There is also no point in checking whether the list can contain the antidiagonal sequence defined by DEF *s* because the solution would not come from *T*.

An antidiagonal string placed in a certain line of the list would have to contain an illegal character to satisfy Cantor's formula:

In Cantor's original version using the characters "m" and "w", the equivalent of ½ could be the letter "w" rotated by 90 degrees:

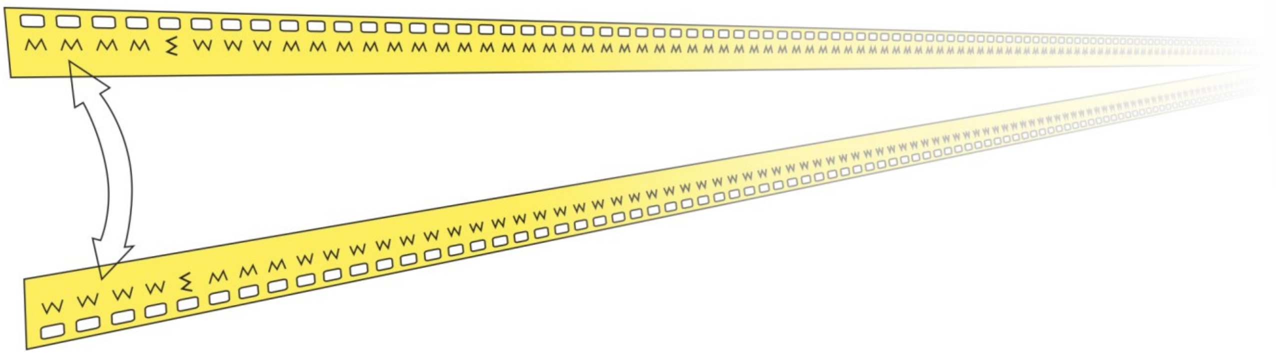


Figure 13. Complementary Cantor sequences with an illegal sign.

A sequence of characters containing a rotated character in the fifth position would satisfy the definitional criteria for the antidiagonal inserted in the fifth line of the list as a mirror image of the diagonal. However, such a sequence would not come from *M*, nor would it consist exclusively of the allowed characters.

The examinations of the functionality of *CDA* have been undertaken with accurately formulated lists comprising permissible character sequences. These observations have been extrapolated to encompass all potential scenarios, predicated on the assumption that the method will be applicable in every instance.

Nevertheless, if a situation arises wherein the antidiagonal definition functions incorrectly for certain lists—meaning it fails to produce an unambiguous sequence composed of permitted characters—then the outcome of this operation cannot be incorporated into the set *T* nor can it be integrated into the list, whether it be the one currently under consideration or any other.

Consequently, the absence of a defective product on the list is not indicative of anything, as by definition, the list is designed solely to be comprised of correct elements.

For the study and comparison to be efficacious across all lists, each comprising an infinite collection of infinite sequences, it is imperative that they be conducted in a manner that employs a finite formula. This formula should define the methodology for generating the successive characters of a sequence and facilitate comparison of their infinite quantities, rather than relying on the direct comparison of characters in the sequences being examined.

This approach underscores a critical distinction between a specific sequence that is articulated solely through an infinite assembly of permissible characters, and a definition—a recipe—that permits the unambiguous generation of an arbitrarily large number of characters. The Cantor Diagonal Argument (CDA), by employing the antidiagonal definition formulated as a universal application for all conceivable, accurately constructed lists, is responsible for generating a sequence that exhibits the equivalent properties as those of the elements included in the list. The theoretically derived, correct sequence could potentially reside within the list, providing the foundation for Cantor's argument that the list is incapable of encompassing it, thereby indicating that the source set comprises a greater multitude of elements than that of the natural numbers.

It is imperative to note that a flawed definition results in the generation of an invalid sequence, precluding the ability to search for such a sequence within a set restricted to sequences comprising only valid characters.

Neither formula is a valid definition: the "definition s_d" is ambiguous, and the "definition s" is antinomian.

Both definitional formulas can be improved, as I will show below.

4.10 Can Cantor's definitions be made more precise?

The error in Cantor's definition consists of building a sequence based on a source list that may contain a given sequence. For non-self-referential lists, such an error does not occur (e.g., lists of rational numbers that generate irrational numbers). Still, for self-referential lists, there is an error in specifying one character in the entire long and infinite chain of characters, located at the diagonal intersection with the sequence's introduction line.

The approach to enhancing the diagonal definition and the antidiagonal definition involves specifying the value of the nth character when the definition is entered into the nth position list.

Improved diagonal $s^0{}_d$ placed on line n of the given list:

$$s^0{}_d = d_1 d_2 d_3 \dots,\quad d_i = a_{ii},\ \text{if } i \neq n$$

$$d_i = 0,\ \text{if } i = n \tag{3}$$

Improved antidiagonal $s^1{}_a$ placed on line m of the given list:

$$s^1{}_a = a_1 a_2 a_3 \dots,\qquad a_i = 1 - a_{ii},\quad \text{if} \neq m$$

$$a_i = 1,\qquad \text{if } i = m \tag{4}$$

Taking into account the possibility of even multiple insertions of the same sequence coming from T into one list (F) but for example, defined by different definitions, the list can contain an infinite number of identical improved diagonal sequences $s^0{}_d$ and an infinite number of improved antidiagonal sequences $s^1{}_a$, but in the latter case each of them will differ not only from the other sequences in the list but will also differ from each other (depending on their position in the list).

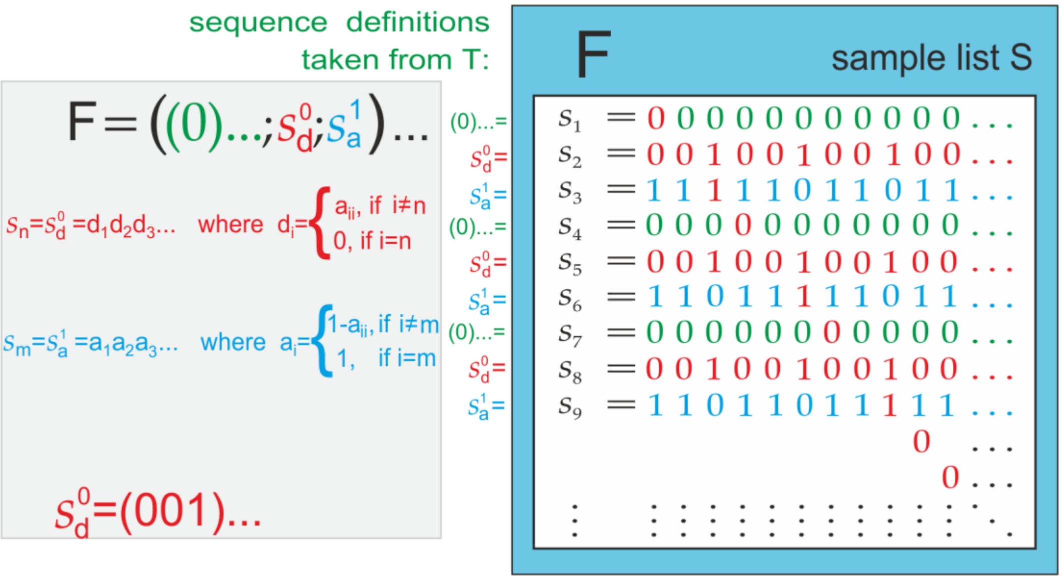


Figure 14. List with corrected diagonals and antidiagonals.

Note that the corrected diagonal is placed in the list F in positions s_2, s_5, and s_8... and it is the same sequence but the corrected antidiagonal from position, for example, s_3 is different from all other rows, including other corrected antidiagonal lines (s_6 or s_9).

In the course of refining the definitions pertaining to the diagonal and antidiagonal, the character designated for entry at the confluence of the input line and the diagonal list line has been identified as "0," annotated with a superscript, within the context of the enhanced diagonal formulation. Conversely, the character "1" has been utilized for the enhanced antidiagonal; however, these roles may be subject to interchangeability. It remains feasible to employ identical characters within these definitions. In each scenario, the enhanced diagonal definition will yield strings that are uniformly identical across multiple lines and congruent with the diagonal sequence. Meanwhile, the strings incorporated into the list and delineated by the enhanced antidiagonal formula will exhibit dissimilarities from the other lines and from the remaining antidiagonal sequences positioned on other lines by a minimum of one character (and, naturally, distinct from the characters of the diagonal sequence, apart from the intersection of the input line with the diagonal).

This is in line with the previously articulated hypothesis **H** in this paper, identifying the definition of s as an antinomy and the complementary definition of s_d as ambiguous.

It is straightforward to identify and select a sequence composed solely of the digits 0 and 1 within a diagonal arrangement also consisting exclusively of zeros and ones. We may be misled into believing that it suffices to generalize observed patterns through elementary definitions while not acknowledging that such definitions can influence the order of characters in particular rows of a list. Moreover, Cantor (and his successors) were fascinated by the construction of string s, which theoretically excludes the possibility of the existence of a string s in a list. It cannot be on this list, but because of an invalid antinominal formula that does not determine the string 0-1 and is accompanied by an ambiguous formula. The paradox within the definition of the string s lies in the fact that this formula does not delineate a string consisting solely of characters

0 and 1, contrary to our anticipations and Cantor's expectations: no acceptable symbol can be positioned at the *k*th place.

Let us recall the **ASSUMPTION** from Cantor's lemma, in which he assured us that for any list taken from *T*, thanks to the antidiagonal definition, it will create a string *s* not included in the list. We can find an infinite number of such lists, and for them, the definition of *s* does not create a chain consisting only of 0s and 1s, and for this reason, it cannot be found in the list or in the set *T*. Therefore, the lemma does not prove the existence of sets of type *T* with cardinality greater than the number of natural numbers.

Theoretically, we could consider a specific array for which we can easily see a diagonal sequence and an anti-diagonal sequence consisting of only these two allowed characters. Can we look for an antidiagonal in the rows of this list? This would be equivalent to the command: "Find an element that differs from itself by at least one character." The point here is not that if such an element is not on the list, it must be in set *T*. The whole procedure is simply defective. It is an antinomy, as we found out when we examined the supplement (dual) to the formula proposed by Cantor and found its ambiguity. Within a binary list consisting exclusively of the digits 0 and 1, a unique sequence is present along the diagonal. Although this unique sequence may be observed in a particular row of the list, it will not conform to the string defined by the antidiagonal according to the definition provided in s_d. In contrast, the sequence will conform to an alternative definition that is distinct and not equivalent to the previously mentioned definition.

Brouwer's hypothesis regarding the potential deficiencies inherent in the definition of an antidiagonal string has been validated.

Also, the related and complementary definition of the diagonal string, which is essentially a modification of Wittgenstein, is flawed by its ambiguity, and the relationship between them is the same as between the known contradictory antinomies and their ambiguous complements (duals).

4.11 *CDA* for real number ℝ

For the most common form of representing *CDA* for real numbers in the range (0,1), it is enough to precede each string with a prefix consisting of the characters "0" and a separator "." - then we will receive a list of real numbers expressed in binary form, a binary Cantor number *s*, the definition of which can be examined, taking into account, of course, the indirect and supplementary definition of the real Cantor number s_d resulting from the highlighted digits on the diagonal of the list without replacing them. All the remarks and considerations concerning infinite binary sequences can be transferred to real numbers in this way.

In the realm of real numbers with decimal expansion, consider the *k*th position in string s_d: a canvas of integers where up to ten can be interwoven without changing any digit. This creates ten distinct numbers, each meeting the criteria of s_d. However, this flexibility limits the parameters of *s*, leading to a paradoxical constraint on implementation.
One of the formulas for a decimal diagonal and antidiagonal number may look like this:

$$s_d = 0.d_1d_2d_3d_4\ldots, \qquad \text{where } d_i = a_{ii}$$

and

$$s = 0.a_1a_2a_3a_4\ldots, \text{ where } a_i = 9 - a_{ii} \text{ or } a_i = 9 - d_i \tag{5}$$

an attempt to place the antidiagonal in the *k* row leads to the equations:
$a_k = 9 - a_{kk}$
$a_k = a_{kk}$
$a_k = 9 - a_k$
$a_k = 4.5 \notin \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

4.12 Summary: *CDA* - the source of antinomy *s* and ambiguity s_d

The assumption cited in the proof of reductio ad absurdum cannot be an antinomy – because the law of excluded middle to antinomy does not apply - and we are dealing here with an antinomy, which results in the invalidity of Cantor's proof and the resulting larger size of the set composed of all subsets of the set of natural numbers. It is false to believe that $f: \mathbb{N} \to P(\mathbb{N})$ cannot be surjective.

The method for identifying flawed definitions and self-referential constructs via their complements in hypothesis H highlights the need to revise the axioms.

The Axiom of Specification should be supplemented with the exclusion of antinomic and ambiguous predicates, and the commonly accepted name of the Axiom of Power should be changed to the Axiom of All Subsets - the existence of a set consisting of all subsets of a given set.

5. Cantor Method for Nested Intervals (MNI)

Jules Richard showed that the set of all texts over an arbitrarily rich and finite alphabet is countable. The list of all texts includes any subsets containing objects with selected features and their definitions (e.g., real numbers, sequences, subsets, strings), but their extraction may encounter qualification problems.

5.1 CC - Cantor Criterion

One of the key points of Cantor's method is the selection of the correct subset of real numbers. Cantor noted that the proper definition of a real number should allow determining any number of digits in its digits expansion (for example, in binary or decimal systems). This principle can be defined as the CC - Cantor's Criterion. If a text under investigation defines a real number, but it is not possible to determine the digits of its expansion using CC, this text cannot be included in the subset R, which contains only texts that satisfy the CC criterion and define real numbers from the set of all texts **T**.
By applying *CC* for real numbers to a list of all texts, we can extract a countable list R that defines real numbers, which by definition need not contain all real numbers from $\mathbb{R}$.

In the (constructive) proof of the uncountability of real numbers by nested intervals[20], Cantor, for any list of real numbers and any interval (a,b), constructs a number defined as

$$a_\infty = lim_{i\to\infty} a_i \text{ or } b_\infty = lim_{i\to\infty} b_i.$$

5.2 PL - Separating Permutation according to *L*

In the rest of the article, we will use Cantor's observation to apply the Cantor Criterion to the analysis of the nested interval method, while eliminating harmful self-reference from the examined list of real numbers. This means that for any fixed real number $L \in \mathbb{R}$, we will try to rearrange the components of the set R in such a way as to obtain a sequence convergent to the number L, using the nested interval method. We will check which ingredients will not be included in the created list (they will remain on the leftovers list). We will examine the limit of convergence and the possibility of its existence in the set R.

Let us take any

$$L \in \mathbb{R} \tag{6}$$

The number L is a real number subject to the Cantor Criterion, and its scientific notation can be represented by numerical expansion in exponential notation with base (positional system) p:

$$L = s * p^k * l_1.l_2l_3l_4 \ldots \tag{7}$$

$$L_n = s * p^k * l_1.l_2l_3 \ldots l_n \tag{8}$$

L_n - a number consisting of the first n significant digits of the number L.
where $s = sgn(L)$- the value of L, p - bases of the positional system (most often binary or decimal), k - exponent for a given number, and l_1 - first significant digit, l_2 - second significant digit, etc.

Let us put

$$\varepsilon = p^{k+1} \tag{9}$$

Construction of the list X_L by induction

Step 1
From the pool of definitions of real numbers from list R, we transfer to the first position of the created list X_L a number of the form: $x_{L1} = a_1 = L_1 - \varepsilon$, and for the second position text $x_{L2} = b_1 = L_1 + \varepsilon$. Now, single-character definitions from the list R will be considered, i.e. those that are defined by one character (e.g., 7, 9, etc.), and if they define numbers from the range (a_1, b_1), they will remain in the list R, but if they are outside this range, will be successively placed on the X_L list as the next elements of this list, i.e. x_{L3}, then x_{L4}, etc. and simultaneously removed from R.

This will create a new list R_{L1} - a list of text definitions of real numbers contained in the interval (a_1, b_1), also containing all texts longer than 1 character.

Step n
From the $R_{L(n-1)}$ list, we transfer the next two numbers denoting the next p-fold narrowing of the range, but with the center

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[20] https://en.wikipedia.org/wiki/Cantor%27s_first_set_theory_article#Second_theorem

determined by the first n digits of the number L, i.e.

$L_n = s * p^k * l_1 . l_2 l_3 l_4 . . . ln$, forming the interval $(a_n, b_n) = (L_n - \varepsilon/(p^{n-1}), L_n + \varepsilon/(p^{n-1}))$.

Next, we will analyze the definitions of the list $R_{L(n-1)}$, which consist of a maximum of n characters (e.g.:8;19; 3.1; etc.), if they fall within the new range, they will remain in the list $R_{L(n-1)}$, and if not, they will be placed in the list X_L and removed from the list $R_{L(n-1)}$.

The list X_L will grow, and the list $R_{L(n-1)}$ will shrink.

A new R_{Ln} list will be created, a list of text definitions of real numbers in the range $(L_n - \varepsilon/(p^{n-1}), L_n + \varepsilon/(p^{n-1}))$, also containing all texts longer than n characters. ***End construction list XL***

In the (considered constructive) proof of the uncountability of real numbers by Nested Interwals Cantor, for any list of real numbers and any interval (a,b), constructs a number defined as

$$a_\infty = lim_{i\to\infty} a_i \text{ or } b_\infty = lim_{i\to\infty} b_i.$$

For n tends to infinity, the limit of the sequence of nested $lim_{i\to\infty}(a_i, b_i)$ intervals formed from the list X_L will, of course, be the number $L \in \mathbb{R}$, which we assumed at the beginning and ensured by the construction of this sequence.

5.3 MNI - Method Nested of Intervals (for *XL*)

Interestingly, what will remain in the list R_{Ln} as n goes to infinity? We can label this boundary list as $R_{L\infty}$.

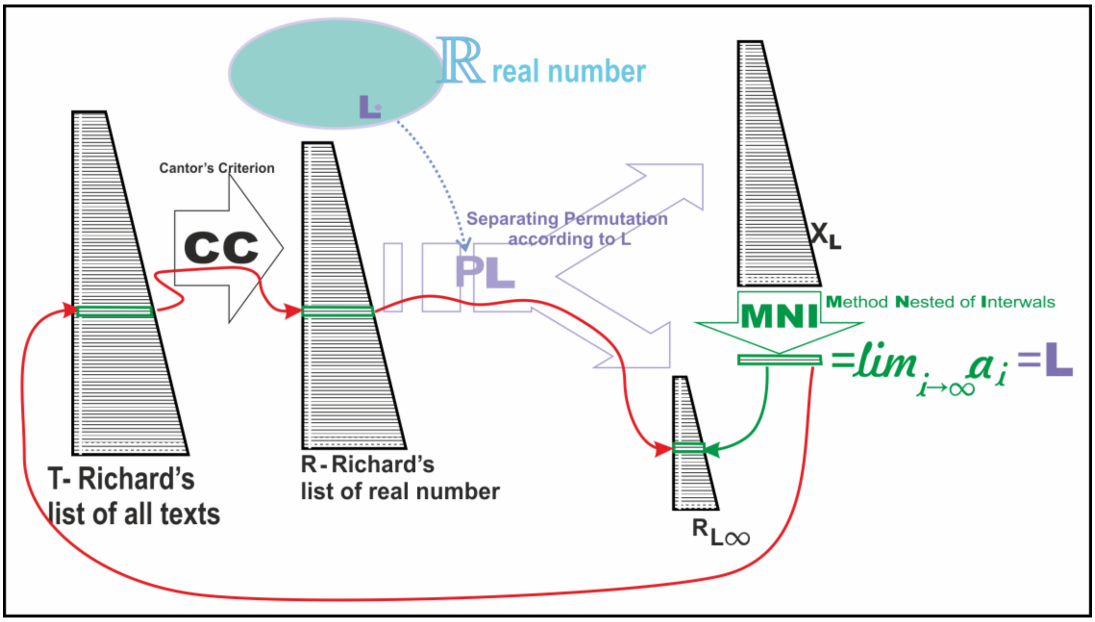

Figure 15. Scheme for depriving the Method Nested of Interval of defective self-reference.

Can any definition of the number $y \in R$ other than L be left on this list?

No, because we can find n such that $\varepsilon/p^n < |y - L|$ and y will be outside all subsequent nested intervals converging to L for steps greater than n.

Can this list be empty? It can contain text that defines L as $a_\infty = lim_{i\to\infty} a_i$, (and this is shown by the green line), but it does not have to. The *MNI* for X_L is constructive and defines a number as text $lim_{i\to\infty ai}$. This text is obviously included in the **T**-list of all Richard's texts, and since Cantor's Criterion is intended to extract from the **T**-list only those texts that we are sure to define real numbers and place them in the R-list, we can join this definition (as well as all similarly derived ones) as correct, which will result in $R_{L\infty}$ being found on the newly created list. (Red line).

For each number L in the set of real numbers, we have shown how to separate all elements of the list R and create a list X_L convergent by the Nested Intervals method to the number L and a complementary list $R_{L\infty}$. Thanks to the definition of convergence obtained by *MNI*, we show that this arbitrarily chosen number from ℝ also has an unambiguous and finite textual definition from R.

5.4 Conclusion from MNI

$\neg(\exists L \in (\mathbb{R} \setminus R))$ so $\mathbb{R} = R$ countable set.

6. Discussion

The procedure used in the previous chapter to remove self-reference from the suspicious nested interval method resulted from the difficulty in finding a dual in the form of a hypodox to *MNI*, which may be a challenge for researchers of hypothesis H. However, the conclusions of this procedure also seem correct and promising. The belief about the greater cardinality of the set of transcendent numbers (otherwise transcendent and supposed to determine the significant predominance of real numbers), based on *CDA* and *MNI*, than the set of natural numbers, is untrue, because for every two transcendent numbers L and K different from each other $L < K$, but located arbitrarily close to each other, we can find such a natural number n, for which the digital expansions of both numbers differ from each other in the nth place: $L_n \neq K_n$, i.e.: $\delta = K_n - L_n > 0$ and between both numbers L and K we can introduce an infinite number of rational numbers in the form $a_k = K_n - \delta/p^k$, where $k \in \mathbb{N}$, p is the basis of the digital system. From the possibility of introducing an infinite number of rational numbers between any two transcendental numbers (and any two other real numbers), one should conclude that there are more rational numbers than transcendental numbers and not the other way around.

In fact, however, we know perfectly well that there is a countable number of rational numbers, and therefore there cannot be more transcendent numbers (and real numbers). In the above reasoning, a similar procedure was used as in the Nested Interval Method, and by using it, we could "prove" that not all rational numbers can be found in any list of real numbers because each of the smaller and smaller intervals in the nested sequence is nonzero and contains an infinite number of rational numbers, and each of these rational numbers can be the limit of the sequence of nested intervals if only we arrange them in the appropriate way on the list used by *MNI*.

Showing the impossibility of establishing a bijection using the chosen method does not prove the non-equivalence of infinite sets because there may be another way of establishing a bijection between them.

A sufficient condition for proving the lack of quantitative superiority of the set under consideration over the set of natural numbers is the existence of a surjection. This was used in "Cantor's proof" to show that the lack of such a surjection for set B proves the impossibility of equivalence. Therefore, if we show a surjection of the natural numbers onto some set, we can conclude that this set cannot be larger than the set of natural numbers.

Now consider Cantor's definition in light of Richard's paradox. Assuming that this definition *N*(antidiagonal), correctly generates a real number for any list of real numbers *E*, and therefore also for an infinite list of textual definitions of real numbers, Richard shows that such a list *E* can be ordered by applying the principle of increasing text length and lexicographic ordering. Consequently, the definition of *N* implies that a real number defined by it with a finite number of characters should be included in this list. However, since it must differ in its numerical expansion from any numbers listed in *E* in the position that the number would occupy, it cannot be included in it. There are many conflicting explanations for why this is merely an apparent paradox. In I. J. Good's discussion [12], the reason is that it is impossible to determine for each textual definition whether a given text correctly defines a real number in a finite number of steps, which makes it impossible to precisely define the list *E* and the number *N* derived from it.

First, note that the list *E* is constructed from a countable list **T**, a list of all possible texts for an arbitrarily large but finite alphabet. The list *E* is constructed by eliminating all texts from **T** that are not definitions of real numbers. Cantor's criterion can be used as a method for eliminating texts that do not define real numbers.

Therefore, such a method exists, and an element of *N* should correctly define a real number. Element *N* should be in the list *E*, while at the same time being different from every element of the list (by definition).

Assuming that the aforementioned definition text accurately specifies the number *N*, it follows that this definition constitutes an element of the countable set of all texts **T**. Consequently, it is associated with a particular natural number through the surjective mapping from the natural numbers onto the set of all texts. This association demonstrates that it cannot be utilized as evidence for the uncountability of the real numbers.

Second, notice that the *CDA* procedure should work for any infinite list containing proper objects. We do not have to consider all proper elements in the list, and the order of appearance in the list is arbitrary. Thus, even without using Cantor's Criterion, we can construct an infinite list of real numbers R1 by starting with Cantor's antidiagonal definition (CA), and for subsequent entries, select from all texts only those that define the reciprocals of the ordinals in the list. The list R1 will be a sublist of Richard's list *E*, and if the antidiagonal definition CA were true, one could correctly determine all digits of the number it defines:

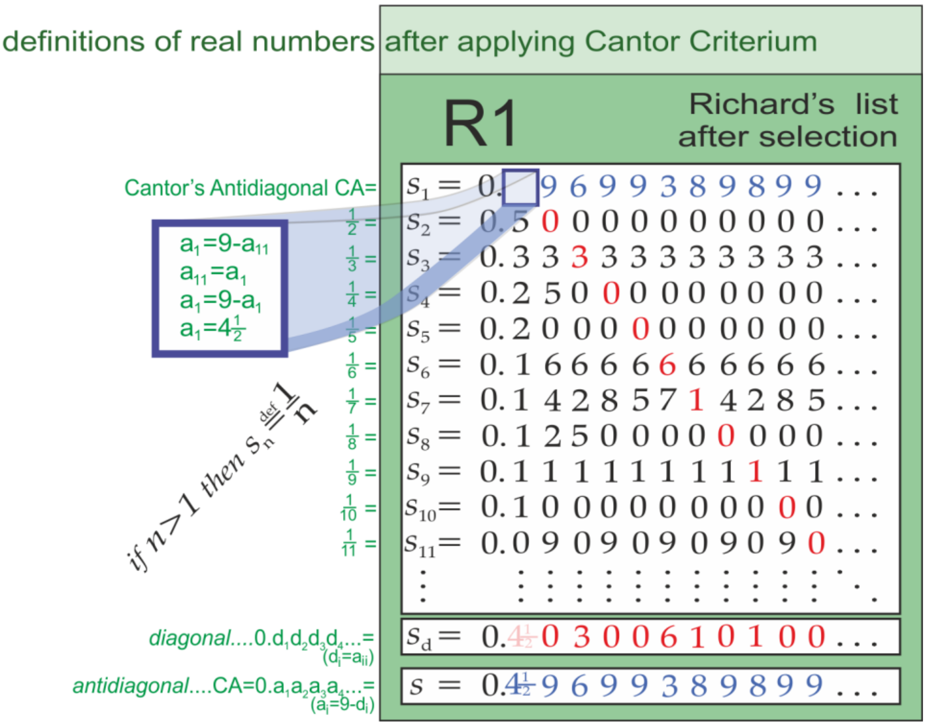

Figure 16. Richard's list with antidiagonal.

There are also other methods of changing signs from diagonal to antidiagonal, which do not use the above formula - you can use any and try unsuccessfully to define the first sign on the list...

What sign can we unambiguously insert into another list if, instead of antidiagonal, we put the complementary definition of Cantor's diagonal CD in the first position?

Let's see what this results in.

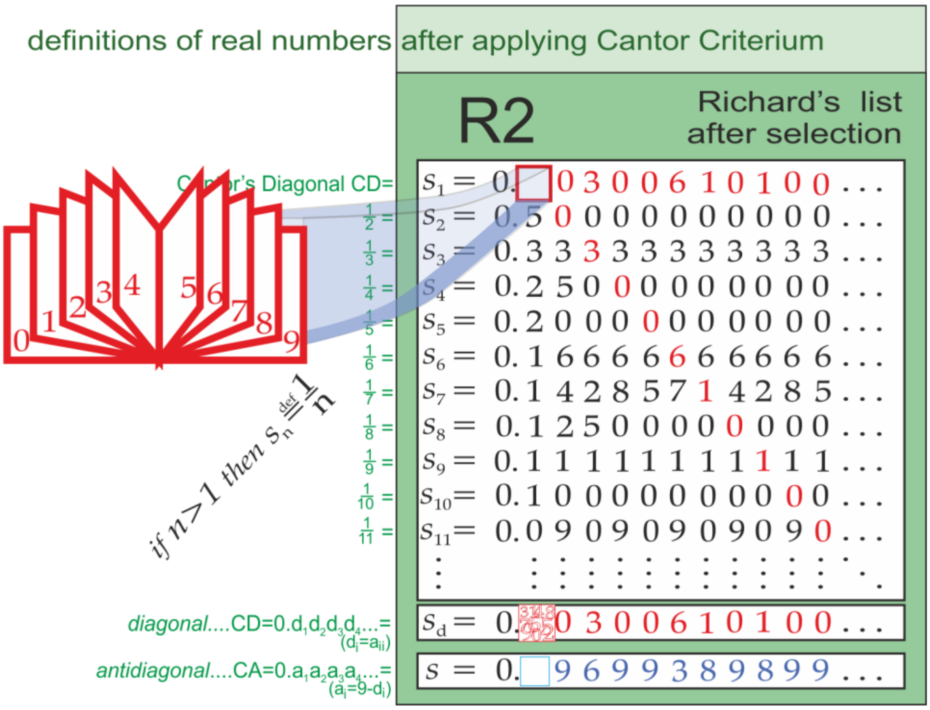

Figure 17. Richard's list with a diagonal.

The above attempts to include the antidiagonal definition or the diagonal definition in Richard’s lists of real numbers end in failure for these definitions, after applying the Cantor criterion to verify whether a given definition is correct. The Cantor criterion rejects incorrect definitions and does not allow them to be included in the list, but, of course, such a list can be created after removing defective definitions:

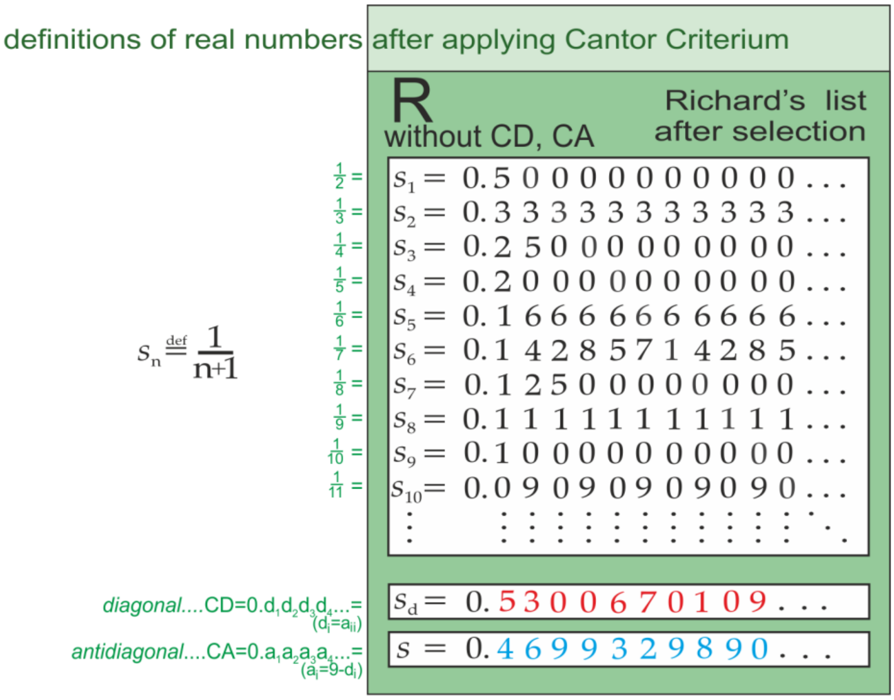

Figure 18. Richard’s list without antidiagonal & diagonal.

Following the elimination of incorrect definitions from the aforementioned list, it becomes possible to compute both the diagonal and antidiagonal sequences, as illustrated.

However, by its very definition, such a list inherently cannot contain an antidiagonal sequence. Thus, verifying its inclusion within the list is superfluous. Moreover, it fails to demonstrate the non-equivalence of set cardinalities, as the attempt to incorporate them within the list discloses their inherent deficiencies.

The Cantor Criterion can eliminate faulty formulations from science in a similar way to the H hypothesis, and their range of applications is similar, but the H hypothesis allows for wider application from various branches of science, not only from mathematics and set theory.

A. Billon’s [4] indication in Richard’s paradox of duality in the form of hypodoxy, characterized by an ambiguous and inadmissible form, also confirms hypothesis H put forward in the article on relations between antinomial self-referential contradictions and their ambiguous complement. It should be remembered that Richard’s paradox arose as a denial of the correctness of the diagonal method, and although an explanation was found that "it is not the definition that is flawed", transferring the problem to the difficulties of interpreting texts and constructing correct lists, neither the above findings nor Billon’s conclusions confirm this explanation - on the contrary, they point to a flaw in Cantor’s antidiagonal definition.

However, since both *CDA* and MNI are the only arguments that are supposed to prove the existence of uncountable sets, i.e. infinite sets with more elements than there are in the set of natural numbers, and their antinomial flaw has been demonstrated, the entire edifice of set theory concerning the existence of uncountable sets loses its basis of existence. The differences in the cardinality of infinite sets, together with the scale of alephs and similar derived concepts, also lose their significance.

I often come across the objection that the diagonal method can only be applied to a fully completed, specified list that does not contain an element that is being created, whereas the antidiagonal method because that is only created as a result of the *CDA* operation on the finished list. If that were the assumption, then there would be no point in checking whether the antidiagonal does not appears in the list. We also remember that it could be defined using another definition, and all equivalent definitions generate the same sequence.

The Pythagoreans already noticed that they could not write down some geometric distances in numerical form. They used only a finite digital notation, and it did not provide such a possibility. However, we can see that these are finite distances (e.g., the circumference of a circle or the diagonal of a square, etc.) and their notation can also be finite, but using additional symbols - not necessarily digital, and then we can easily perform certain mathematical operations on them, such as square $\sqrt{2}$ (result = 2 - EXACTLY!) or cube it (result = $2\sqrt{2}$), i.e. write down numbers, operations, results using additional precisely specified symbols.

However, the path proposed by Cantor, leading to the use of an infinite number of digits to define irrational numbers, is not the best for several reasons.

- inability to write out all the digits of the notation (we lack time)
- inability to perform calculations on them
- inability to compare two slightly different numbers due to the inability to read all the places - having two very long chains of digits at our disposal, broken off by three dots, we cannot answer whether these chains are equal or different.
  - We can calculate any number of digits of their numerical expansion from the knowledge of irrational numbers, defined by their definitions (e.g., geometric distances or others), BUT NOT BACKWARDS!!!! We cannot deduce from the infinite digital notation of a real number whether it could be generated using a finite verbal definition, and this is what happens in the cases known to us.

If numbers are to be used for counting, then we can accept their symbolisms like π, $arctg0.3$, $\sqrt{7}$, $(2 + sin3)$, e^3, etc., and perform certain and precise calculations on them.

Mathematicians who accept Cantor's proofs of the existence of uncountable sets, along with the understanding that all the texts, definitions, proofs, and reasonings are in the countable set of all the texts on Richard's list—with a certain degree of resignation, it seems to me—admit that much of mathematics defies precise description.

It can only remain an abstract concept, like the powers attributed to deities, superluminal travel, and phenomena occurring beneath the event horizon of a black hole.

That part of set theory that describes the consequences of the existence of uncountable sets is something of an act of faith on the part of its practitioners, because what they describe, trying to clothe their texts in logical and sophisticated patterns, is only a semblance of truth that seemed unfalsifiable in Popper's terms.

After all, we can believe in the existence of ideal infinity, use it in differential calculus, perspective painting, where parallel lines intersect on the horizon at infinity, and we can discover and name any number, even a huge one, tame it - all of this is there for us to discover and is not hidden behind the impassable horizon invented by Cantor.

Accepting Cantor's flawed proofs opened the gates to a fantastic mathematical paradise, full of colorful bubbles, but having nothing to do with reality, even the imaginary = Platonic one. The current situation is perfectly described by the words[21] of Prof. Bogdan Dembiński 22 [49:11]: (Dembiński, 2016):

> *The combinatorial mind can create an infinite number of models. Of course, with this ability, we can stay at this level of building various infinite models and recognize that here we are fulfilling our task by creating these models. But we also have the order of the universe. We have the cosmic structure that we observe; all these regularities, all the rules, are something that is not an arbitrary state and that we can influence.*
> *I would say this: "If I have a physical law, then it not only decides that something behaves in a certain way and not another and explains how, but it is what I must reach for if I want to understand the situation that I am describing. I cannot stay only in the area of these models."*
> *How does mathematics relate to ideas? Well, according to Plato [in my summary]:*
>
> > *Ideas are some equivalent of the laws of nature that organize the cosmic order. They are independent of our decision: they do not come into being, do not last, and do not perish. They simply are! [...] We can get to them in different ways: by analyzing the way the cosmic order exists, but we can also get to them by creating intellectual models or mathematical models. However, we must remember that these mathematical models cannot replace these laws and regularities. They are not the same as these regularities. [...]*
> > *We create languages, and this is our creation - in this sense, mathematics is created, but it is not a situation in which language replaces reality.*

Because scientific reality does not tolerate contradictions, even in those created in minds, according to Shand.

7. Conclusions

In mathematics, it is unthinkable to accept contradictions, and Cantor's proofs appear as antinomies. The Axiom of Specification ought to be augmented by the exclusion of antinomial, and ambiguous formulas. There are no infinite uncountable sets. They all have the cardinality of the set of natural numbers.

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[21] https://www.youtube.com/watch?v=ydXm-Up77SM&t=2s

The CH-continuum hypothesis is true because the cardinality of natural numbers and real numbers is the same and there is no set with intermediate cardinality.

Demonstrating the flawed nature of Cantor's proofs allows us to emerge from this gloomy fog and restores to mathematicians the possibility of discovering and naming elements of reality, instead of self-referentially inventing such UFOs, i.e. unnameable fantasy objects that cannot be themselves.

Mathematicians!

There is a lot of work ahead of you:

- accepting changes to the axioms, especially the Specification Axiom, to eliminate antinomies, and ambiguous.
- checking the validity of other self-referential arguments (including Turing's STOP problem and Gödel's theorem); - cleaning up set theory from errors.

Acknowledgments

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