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“Fermat’S Great Theorem”

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Abstract

The work contains a complete proof of the "Fermat’s great theorem" farm elementary methods, which is approved by well-known scholars in the field of number theory, and is intended for all lovers of mathematics. “Fermat’s Great Theorem”, there do not exist integral numbers x, y, z different from zero, for which:

$$x^n + y^n = z^n, \tag{1}$$

where $n > 2$ (it is well known that at $n = 2$ such numbers exist).
From equation 1 should be

$$\frac{x+y-z}{n} = 2\lambda > 0,$$

where λ a positive integer

$$x + y - z = 2\lambda n.$$

As in any set of natural numbers there exists the smallest number, among all such solutions there exists a solution x, y, z with the smallest value of λ. Let us examine this solution in more detail.
By entering these designations in the equation (1), we shall get

$$(2\lambda n + x_1)^n + (2\lambda n + y_1)^n = (2\lambda n + x_1 + y_1)^n.$$

Keywords

Number Theory, Theorem, Complete Proof.

1. Introduction

Let us assume that there exists a solution of equation (1) in whole, different from zero, numbers. It is obvious that, without loss of generality, we can consider that it consists of pairwise positive co-primes. Further, it is obvious that if Fermat’s theorem is correct for *n* index, then it automatically turns out to be correct and for any index, multiple of *n* because the equation $u^{an} + v^{an} = w^{an}$ has an integral solution *u*, *v*, *w*, than the equation (1) will have an integral solution u^a, v^a, w^a. That is why it is enough to prove Fermat’s theorem for $n=4$ (this was done by Fermat himself) and for $n\geq3$ is an arbitrary prime number. It may be considered that $x < y < z$ *or* $y < x < z$ (which is not) (If $x = y$, then $2x^n = z^n$ or $\left(\frac{z}{x}\right)^n = 2$; $\frac{z}{x} > 1$, is a rational number. It is known that there does not exist rational number the n^{th} degree of which is equal to 2.

From equation (1) should be

$$(x^n - x) + (y^n - y) = z^n - (x + y), \quad x + y > z.$$

Due to Fermat's little theorem (formulation Euler's [1]), we have

$$x^n \equiv x \bmod n;\quad y^n \equiv y \bmod n \Rightarrow z^n \equiv x + y \bmod n$$

On the other hand $z^n \equiv z \bmod n \Rightarrow x + y \equiv z \bmod n$, $x + y - z$ – even number, n-odd number $\frac{x+y-z}{n} = 2\lambda > 0$,

where λ a positive integer

$$x + y - z = 2\lambda n. \tag{2}$$

As in any set of natural numbers there exists the smallest number, among all such solutions there exists a solution *x*, *y*, *z* with the smallest value of λ. Let us examine this solution in more detail.

From expression (2) is $x = 2\lambda n + (z - y) = 2\lambda n + x_1$, where $x_1 = z - y > 0$, analogously, $y = 2\lambda n + y_1$, where $y_1 = z - x > 0$; and, consequently, $z = 2\lambda n + x_1 + y_1$.

By entering these designations in the equation (1), we shall get

$$(2\lambda n + x_1)^n + (2\lambda n + y_1)^n = (2\lambda n + x_1 + y_1)^n. \tag{3}$$

In accordance with the Newton's binomial formula

$$\begin{aligned}(2\lambda n)^n &+ C_n^1 (2\lambda n)^{n-1}\left[(x_1 + y_1) - (x_1 + y_1)\right] + \\ &+ C_n^2 (2\lambda n)^{n-2}\left[(x_1^2 + y_1^2) - (x_1 + y_1)^2\right] + \cdots + \\ &+ C_n^{m-1}(2\lambda n)\left[(x_1^{n-1} + y_1^{n-1}) - (x_1 + y_1)^{n-1}\right] + \\ &+ (x_1^n + y_1^n) - (x_1 + y_1)^n = 0,\end{aligned}$$

or

$$(2\lambda n)^n = x_1 y_1 \begin{bmatrix} 2C_n^2 (2\lambda n)^{n-2} + 3C_n^3 (2\lambda n)^{n-3}(x_1 + y_1) + \cdots + \\ + (C_n^1 x_1^{n-2} + C_n^2 x_1^{n-3} y_1 + \cdots + \\ + C_n^{n-2} x_1 y_1^{n-3} + C_n^{n-1} y_1^{n-2}) \end{bmatrix}. \tag{4}$$

Let us designate the expression given in square brackets in equation (4) by α:

$$\begin{aligned}\alpha = 2C_n^2 (2\lambda n)^{n-2} &+ 3C_n^3 (2\lambda n)^{n-3}(x_1 + y_1) + \cdots + \\ + C_n^{n-1}(2\lambda n)&(C_{n-1}^1 x_1^{n-3} + \cdots + C_{n-1}^{n-2} y_1^{n-3}) + \\ + (C_n^1 x_1^{n-2} &+ C_n^2 x_1^{n-3} y_1 + \cdots + C_n^{n-2} x_1 y_1^{n-3} + C_n^{n-1} y_1^{n-2}).\end{aligned} \tag{5}$$

Because all binomial coefficients divisible by *n*, so we can assume that *α* is always divisible by *n*, i.e., we have

$$(2\lambda n)^n = x_1 y_1 a$$

2. Methods

2.1. <u>The first case of Fermat's theorem</u>. None of the numbers *x*, *y*, *z* are divided by *n*, i.e.

$$(x_1, n) = 1; (y_1, n) = 1; (x_1 + y_1, n) = 1.$$

In expressions (5) x_1, y_1 and *α* co prime, really from the equation (4) it is obvious that λn in expansion contains all the prime multipliers that enter into expansion into prime multiplies x_1 and y_1, therefore x_1 and y_1 are co-primes, as *x* and *y* are co-primes. If x_1 and α will common factor, then from equation (5) shows that guidance common factor share of all members in equation (4), except for one $C_n^{n-1} y_1^{n-2} = n \cdot y_1^{n-2}$. Similarly, if y_1 and α will be common factor, then from equation (5) shows that guidance common factor share of all members in equation (4), except for one $C_n^1 x_1^{n-2} = n \cdot x_1^{n-2}$. Similarly it is proved for Abel's formulas [1].

Separately consider the first case Fermat's Theorem for $n = 3$

$$x^3 + y^3 = z^3.$$

From expression (3) is

$$x = 3\lambda + x_1, \quad y = 3\lambda + y_1, \quad z = 3\lambda + x_1 + y_1,$$

Where $(x_1,3)=1; (y_1,3)=1; (x_1+y_1,3)=1$, λ - even number,

$$(3\lambda + x_1)^3 + (3\lambda + y_1)^3 = (3\lambda + x_1 + y_1)^3,$$

or

$$(3\lambda)^3 = x_1 y_1 [2 \cdot 3(3\lambda) + 3(x_1 + y_1)].$$

none of the numbers *x*, *y*, *z* are divided by 3, i.e. x_1, y_1 and $x_1 + y_1$ is not divided by 3.

$$3\lambda^3 = x_1 y_1 \left[2\lambda + \frac{x_1 + y_1}{3} \right],$$

i. e. $x_1 + y_1$ is divided by 3, we will get violation [2].

Based on this first occasion Fermat's theorem proving enough for $n \geq 5$.

A basic role in all the arguments related to Fermat's theorem, is played by the following lemma: If the product of two co-prime natural numbers is an nth degree, then each of the multiplicand will also be an nth degree [3]. As in expressions (5) x_1, y_1 and α co primes and consequently, therefore in equality

$$(2\lambda n)^n = x_1 y_1 a, \tag{6}$$

The first case: *x* or *y* are even number (which is not), say *x* - even number in accordance with the lemma, there exist such integers x_2, y_2 and γ that

$$x_1 = (2x_2)^n, \quad y_1 = y_2^n, \quad a = \gamma^n, \quad x_2 \neq y_2, \tag{7}$$

where $(x_2,n)=1$, $(y_2,n)=1$, $(x_2,\gamma)=1$, $(y_2,\gamma)=1$. From (5) shows that the values α depends on the λ values, i.e. γ - too is a function λ, $\gamma = \gamma(\lambda)$.

Substituting the values (7) in equation (6), we shall get

$$(2\lambda n)^n = (2x_2)^n \cdot y_2^n \cdot \gamma^n,$$

or

$$2\lambda n = 2x_2 \cdot y_2 \cdot \gamma, \quad \lambda n = x_2 \cdot y_2 \cdot \gamma,$$

where $(x_2 \cdot y_2, n) = 1$, γ - is divided into n, i.e. have $\lambda = x_2 \cdot y_2 \cdot (\gamma / n)$, or $1/(x_2 \cdot y_2) = (\gamma/n)/\ \lambda$.

Equality true then and only then, when $(\gamma/n) = 1$ and $\lambda = x_2 \cdot y_2$, it is clear that in this case λ - will be the smallest, or

$$(\gamma / n) = p, \quad \lambda = p.(x_2 \cdot y_2) \tag{8}$$

where *n* is a prime number, and $(n, x_2 y_2) = 1$, $(\gamma, x_2 y_2) = 1$, $p > 1$- any whole positive number. It is necessary to consider two case: $P = 1$ and $p > 1$.

From equation (8) implies that

$$\begin{cases} \lambda = x_2 y_2; \\ \gamma = n \end{cases} \quad \text{and} \quad \begin{cases} \lambda_1 = p \cdot x_2 y_2; \\ \gamma(\lambda_1) = p \cdot n, \end{cases},$$

this clearly corresponds to the fact that we approved, if x, y, *and* z satisfy equation (1), then it will satisfy *px, py and pz* , in this case, we have $\dfrac{px + py - pz}{2n} = p\dfrac{x + y - z}{2n} = p\ \lambda = \lambda_1 = p \cdot x_2 y_2$, accordingly $\gamma(\lambda_1) = p \cdot n$, As the primitive solution we are looking for with the smallest integer λ, when $p = 1$, from formula (8) it is obvious that the least integers $\lambda = x_2 y_2$ and $\gamma = n$ which satisfy the expression (4), are equal to

$$n^n = 2C_n^2(2x_2 \cdot y_2 \cdot n)^{n-2} + 3C_n^3(2x_2 \cdot y_2 \cdot n)^{n-3} \times \left[(2x_2)^n + y_2^n\right] + \ldots + \left[C_n^1(2^n \cdot x_2^n)^{n-2} + \ldots + C_n^{n-1}(y_2^{n-2})\right], \qquad (8')$$

if p >1, from formula (8') is realized at

$$p^n n^n = p^n\left\{2C_n^2(2x_2 \cdot y_2 \cdot n)^{n-2} + 3C_n^3(2x_2 \cdot y_2 \cdot n)^{n-3}\left((2x_2)^n + (y_2^n)\right) + \ldots + \left[C_n^1\left(2^n x_2^n\right)^{n-2} + \ldots + C_n^{n-1}(y_2^n)^{n-2}\right]\right\} \qquad (9)$$

Thus, from formula (8) it is obvious that the least integers λ and γ, which satisfy the expression (4), are equal to $\lambda = x_2 y_2$, $\gamma = n$, when x or y are even number (which is not), say x - even number. From equation (7) implies that

$$x_1 = (2x_2)^n, \quad 2\lambda n = 2x_2 y_2 \gamma,$$

or

$$\lambda \ n = x_2 y_2 \gamma, \quad \lambda = x_2 y_2, \quad \gamma = n,$$

primitive solution has the form

$$x = 2x_2 y_2 n + (2x_2)^n,$$
$$y = 2x_2 y_2 n + y_2^n,$$
$$z = 2x_2 y_2 n + (2x_2)^n + y_2^n.$$

In accordance with equation (4), if the right side takes just the first member, we have

$$(2x_2 y_2 n)^n > n(n-1)(2x_2 y_2 n)^{n-2}(2x_2 y_2)^n,$$

or

$$\frac{n}{n-1} > (2x_2 y_2)^{n-2},$$

the inequality is wrong, as at $n \ge 5$, $1 < \frac{n}{n-1} < 2$, $(2x_2 y_2)^{n-2} > 2$.

The second case: x and y are odd numbers, z is even, i.e. x_1 and y_1 are odd numbers, $x_1 + y_1$ is even $x_1 = x_2^n$, $y_1 = y_2^n$, $\alpha = (2\gamma)^n$, where - γ a positive integer, x_2, y_2 are odd numbers,

$$2\lambda \ n = 2x_2 \ y_2 \ \gamma, \ \lambda = x_2 \ y_2 \ \gamma = n,$$

primitive solution has the form

$$x = 2x_2 y_2 n + x_2^n,$$
$$y = 2x_2 y_2 n + y_2^n,$$
$$z = 2x_2 y_2 n + x_2^n + y_2^n$$

In accordance with equation (4) we have

$$(2x_2 y_2 n)^n > n(n-1)(2x_2 y_2 n)^{n-2}(x_2^n y_2^n),$$

or

$$2^2 \frac{n}{n-1} > x_2^{n-2} y_2^{n-2}, \quad n \ge 5.$$

$x_2 \ne y_2$ is an odd number, i.e. $x_2 y_2 \ge 3$, or $\frac{n}{n-1} > \frac{3^{n-2}}{4}$, the inequality is wrong, as at $n \ge 5$.

2.2. <u>The second case of Fermat's theorem</u>. Even one of the numbers *x, y, z* is divided by *n*. If *z* is divided by *n*, then x_1+y_1 also is divided by *n*, at that $(x_1,n)=1$, $(y_1,n)=1$. Analogously, we get that $(x_1,y_1)=1$, $(x_1,a)=1$, $(y_1,a)=1$. In this case the proof will not change.

Let us assume that *x* or *y* is divided by *n*, the proof in both cases will be identical. Without loss of generality, we can consider that *x* is divided by *n*, i. e. x_1 is divided by *n*. As *a* is also divided by *n*, then x_1 and *a* will not be co-primes. Similarly to the first case, y_1 and x_1 are co-primes. Besides, from equation (5) it is obvious that y_1 and *a* in this case too will be co-primes.

The first case: *x* is an even number, *y* and *z* are odd numbers, $(y_1, a)=1$, $(y_1, x_1a)=1$ and consequently, $(2\lambda n)^n=(x_1a)y_1$, where is $y_1=y_2^n$, $x_1a=2^nq^n$, $(2\lambda n)^n=y_2^n2^nq^n \Rightarrow \lambda n=y_2q$, where *q* - a positive integer.

From (5) shows that a common divisor x_1 and α is the only *n*, as all members are obliged to is divided into n^2, member $C_n^{n-1}y_1^{n-2}=n\cdot y_1^{n-2}$ - divided only *n*, the rest are all factors that differ by from n, at the same time, divide the x_1, y_1 и $(2\lambda n)$, this is impossible, because *x* and *y* mutually co-primes number, i.e.

$$\left(\frac{x_1}{n},\alpha n\right)=1\Rightarrow\frac{x_1}{n}(\alpha n)=2^nq^n;$$

$$\frac{x_1}{n}=2^nx_2^n,\ \alpha n=\gamma^n,\ q=x_2\gamma,\ \lambda n=x_2y_2\gamma,\ (x_2y_2,\gamma)=1,\ (x_2y_2,n)=1. \qquad (10)$$

As in the first case, analogously we will get that equation (10) is satisfied at the smallest values of λ and γ, $\lambda=x_2y_2$, $\gamma=n$, equation (10) is satisfied also at $\lambda_1=px_2y_2$, $\gamma(\lambda_1)=pn$, here $p>1$, whole number. Thus, we get:

$$\lambda=x_2\cdot y_2,\ \gamma=n,\ \alpha n=\gamma^n=n^n$$
$$n^n=n\left[2C_n^2(2x_2y_2n)^{n-2}+3C_n^3(2x_2y_2n)^{n-3}(x_1+y_1)+\ldots+\right.$$
$$\left.+(C_n^1x_1^{n-2}+\ldots+C_n^{n-1}y_1^{n-2})\right],$$

If p >1, from formula (5) is realized at

$$p^nn^n=p^n\{n\left[2C_n^2(2x_2y_2n)^{n-2}+3C_n^3(2x_2y_2n)^{n-3}(x_1+y_1)+\ldots+\right.$$
$$\left.+(C_n^1x_1^{n-2}+\ldots+C_n^{n-1}y_1^{n-2})\right]\}$$

As in the first case, analogously we will get that equation (10) is satisfied at the smallest values of λ and γ, $\lambda=x_2y_2$, $\gamma=n$, when *x* is an even number, *y* and *z* are odd numbers $\frac{x_1}{n}=(2x_2)^n$, $y_1=y_2^n$, $\alpha n=\gamma^n$, primitive solution has the form

$$x=2x_2y_2n+n(2x_2)^n,$$
$$y=2x_2y_2n+y_2^n,$$
$$z=2x_2y_2n+n(2x_2)^n+y_2^n.$$

In accordance with equation (4) we have

$$(2x_2y_2n)^n>n(2x_2)^ny_2^nn(n-1)(2x_2y_2n)^{n-2},$$

or

$$1>(n-1)(2x_2y_2)^{n-2},$$

at $n\geq 3$, the inequality is wrong.

The second case: x and y are odd numbers, z is an even number $\frac{x_1}{n} = (x_2)^n$, $y_1 = y_2^n$, $(\alpha n) = (2\gamma)^n$, primitive solution has the form

$$x = 2x_2y_2n + nx_2^n, \quad y = 2x_2y_2n + y_2^n, \quad z = 2x_2y_2n + nx_2^n + y_2^n.$$

or

$$(2x_2y_2n)^n > nx_2^n y_2^n n(n-1)(2x_2y_2n)^{n-2},$$

$$2^2 > (n-1)x_2^{n-2}y_2^{n-2},$$

$x_2 \neq y_2$ is an odd number at the minimum $x_2\ y_2 \geq 3$, at $n \geq 3$, n-1 ≥ 2, in this case the inequality is wrong.

The third case: x is an odd number, y – even number, z – odd number $\frac{x_1}{n} = x_2^n$, $y_1 = (2y_2)^n$, $\alpha n = \gamma^n$, primitive solution has the form

$$x = 2x_2y_2n + nx_2^n, \quad y = 2x_2y_2n + (2y_2)^n, \quad z = 2x_2y_2n + nx_2^n + (2y_2)^n,$$

In accordance with equation (4) we have

$$(2x_2y_2n)^n > nx_2^n(2y_2)^n n(n-1)(2x_2y_2n)^{n-2},$$

$$1 > (n-1)(2x_2y_2)^{n-2},$$

the inequality is wrong at $n \geq 3$.

2.3. Fermat's theorem for indicator 4, will prove a more general statement, equation

$$x^4 + y^4 = z^2 \quad (11)$$

has no solutions in non-zero integers numbers.

Suppose that the solution to the equation (11) in non-zero integers numbers exist. Without losing generality, we can assume that it consists of pair wise co prime positive integers. Consider the decision x, y, z with the lowest z more closely. We assume that an even number x does not restrict the generality of this proposal.

$$(x^2).^2 + (y^2).^2 = z^2 ;$$

number x^2, y^2, z^2 positive and mutually are simple, and the number of x^2 even-odd, the famous lemma there are co prime (m) and (n) ($n < m$) of different parity, that $x^2 = 2\,m\,n$, $y^2 = m^2 - n^2$, $z = m^2 + n^2$.

If $m = 2k$ and $n = 2p$+1, $y^2 = 4(k^2 - p^2 - p - 1) + 3$, that is impossible, since any odd square must be of the form 4 (k) + 1. Consequently, the number of (m) odd, and the number of (n) even-odd. (n) = 2 q, then $x^2 = 4\ m\ q$ or $m\ q = (x/2)$ where (m , q) = 1.

Similarly, we have $m = d^2$, $q = t^2$ where (d) and t –co prime two positive integers

$$y^2 = m^2 - n^2 = (d^2).^2 - (2t^2).^2$$

or

$$(2t^2).^2 + y^2 = (d^2).^2$$

If you apply the lemma, we obtain

$$2t^2 = 2ab, \ y^2 = a^2 - b^2, \ d^2 = a^2 + b^2,$$

Where is a and b (a < b) co prime different parity and t^2 =ab, a $= x_1^2$, b $= y_1^2$, have $d^2 = (x_1^2\ ),^2 + (y_1^2),^2$ or $x_1^4 + y_1^4 = d^2$, this means that the number x_1^2, y_1^2, d represent primitive equation (10), choice Solutions x, y, z must be inequality $d \geq z$, i. e. $d^2 \geq$ z or $m \geq m^2 + n^2$, an absurd inequality.

Accordingly, the suggestion about existence of equation (1) of integral solution results in contradiction.

3. Conclusion

Now we can directly check the fairness of the proposed method of proof of the "Fermat's great theorem" for example the equation

$$x^2 + y^2 = z^2$$

As you know, the primitive equation has the form

$$x = 2mq, y = m^2 - q^2, z = m^2 + q^2$$

where is $q \, and \, m \; (q < m)$ -co prime positive numbers different parity.
Using the above method proof of the "Fermat's great theorem", have

$$x + y - z = 2q\,(m-q)\;,\;\; \frac{x+y-z}{2} = \boldsymbol{\lambda} = q\,(m-q),$$

$$x = 2\lambda + (z-y) = 2\lambda + x_1, \;\; x_1 = z - y > 0 \;,$$

$$y = 2\lambda + (z-x) = 2\lambda + y_1, \;\; y_1 = z - x > 0,$$

$$z = 2\lambda + x_1 + y_1,$$

or

$$(2\lambda + x_1)^2 + (2\lambda + y_1)^2 = (2\lambda + x_1 + y_1)^2 \qquad (12)$$

where is $x_1 = 2q^2,\; y_1 = (m-q)^2$
From equation (12), we have

$$(2\lambda)^2 = x_1 y_1 \alpha,\; \alpha = 2,$$

$$(2\lambda)^2 = \left(\frac{x_1}{2}\right) y_1 (2\alpha),$$

as in the general case

$$\frac{x_1}{2} = x_2^2 = q^2,\; y_1 = y_2^2 = (m-q)^2,\; (2\alpha) = \gamma^2 = 2^2,$$

or

$$(2\lambda)^2 = x_2^2 y_2^2 \gamma^2,\; 2\lambda = x_2 y_2 \gamma,$$

where is $x_2 = q,\;\; y_2 = (m-q),\; \gamma = 2$, or, $\lambda = x_2 y_2 = q\,(m-q),\; \gamma = 2.$

If $\lambda = p x_2 y_2 = p\,q\,(m-q),\; \gamma = 2p$, then along a primitive solution *x, y* and *z* among us will be the decision of *p x, p y* and *p z*.

As in the general case ($n > 2$), the smallest integers λ and γ, which satisfies equations (3) and (5) $\lambda = x_2 y_2$; $\gamma = n$. In this case, we approved the decision, equation (1) has been *x, y* and *z* (co prime positive numbers). When $\lambda_1 = p x_2 y_2$, $\gamma_1 = pn$, decision, respectively, were *px, py* and *pz* .

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