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Generation of Fractal Vessel Structure Functions by Using the Lindenmayer System

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Abstract

A real vessel structure has a complicated geometry that is difficult to define. In this study, we aim to model a vessel structure using the basic principles of the Lindenmayer system. The Lindenmayer system, also known as L-system presents an iterative formula to produce a set of rules for fractal structures requiring complicated and versatile rules. In this paper, we present how we have developed an agent based fractal vessel model using L-system. The geometry of fractal vessel structure is created by using the string rewriting technique. The alphabet symbol codes produced according to the rules of the L-system are visualized by using the turtle graphic algorithm. At the end of this study, we provide the simulation of turtle graphic algorithm in the Repast Simphony framework.

Keywords

Fractal, L-system, Turtle graphic algorithm, Vessel model.

1. Introduction

The nature which includes dynamic systems has simple and/or complex patterns in terms of geometric structure. When these patterns are examined deeply, it is observed that they have simple structures in the core. Fractals help to mathematically explain these patterns which emerge from simple to complex structures. Fractal analyses of diverse systems have been discussed for several years. Fractals explain the mathematical descriptions of many natural phenomena that occur on earth such as lightning, snowflake, rivers and fjords, mountain ranges, trees and leave, living organisms, and etc. [1, 2]. In this study, we discuss with some similarities between structures of blood vessels and branching of plants which are the best examples of fractals. Therefore, we offer an agent based model of branching blood vessels which stems from the fractal vessel structure functions by using the Lindenmayer system. In our agent based blood vessel model consists of a large set of interacting vessel agents [3]. Agent is a software entity which refers to an individual or a system component. Agents perform actions autonomously and interact with other agents in the simulation environment. Each agent has its own characteristics and behaviors depending on the rules set by the programmer [4, 5, 6]. In our previous study [3] which refers to our agent based blood vessel model, we have simulated two types of the vessel models. One of them is a single vessel model without branching and the other one is a bifurcation vessel model. In the modeling of the bifurcation vessel, we have used connectors (the points at which branching vessels start) to implement branching functions of the vessel. However, those connectors cannot create new vessel branches in the model automatically unless we define new branches in the program. Especially, generation of new vessel branches is very complicated when the depth (increasing the number of generation) of the branching vessel increases since the increasing number of connectors makes harder compute the cardiovascular parameters and interact with other vessel agents. In this study, we improve our agent based blood vessel model described in [3] by duplicating vessel branches via Lindenmayer system formalism. We simplify the branching function of the connectors in terms of messaging between agents and computation of parameters by using the string rewriting technique.

This paper is organized into 4 parts. The first part offers a brief account of fractals. The second part provides a detailed account of Lindenmayer system, including components to create the model. The third part presents fractal vessel structure functions which we offer to agent based blood vessel model. And the last part summarizes this study including its limitations and recommendations for future research directions.

2. Fractals

Fractal is a concept introduced by the mathematician Beneoit B. Mandelbrot in 1975. It is derived from the Latin word "fractus" which means broken, fragmented [1, 2]. Fractals are patterns, but every pattern may not be a fractal. The definition of fractal generally refers to structures formed by proportionally shrunk or enlarged models. Fractal structures are observed in many systems in nature. Tree branches, bronchi in our lungs, some vegetables (cauliflower, broccoli, etc.), sunflower, lightning, river beds in the valleys may be considered as examples of fractals. Especially, in the physiology due to the fractal structure of the human vascular system, as the vascular surface area is enlarged; the vessels are able to withstand the blood pressure of the heart [7]. Therefore, many sciences benefit from this model. The living systems in the nature are dynamic and display nonlinear behaviors. It is difficult to model and calculate these systems' behaviors mathematically. Fractal geometry can explain these behaviors through simple rules [8, 9].
The fractal model has an iterative structure that takes its output as an input by repeating itself. It examines the figures that are shrinking or growing forever by repeating itself. Irregular details or patterns are repeated on increasingly reducing scales, and this can be recursive in completely abstract objects, or self-similarity can be observed when any part of the object is enlarged [1, 10]. It is possible to observe fractal curves in any system in nature or in the details of the object. The Von Koch Snowflake is the well-known example of the fractal structures illustrated in Figure 1 [10].

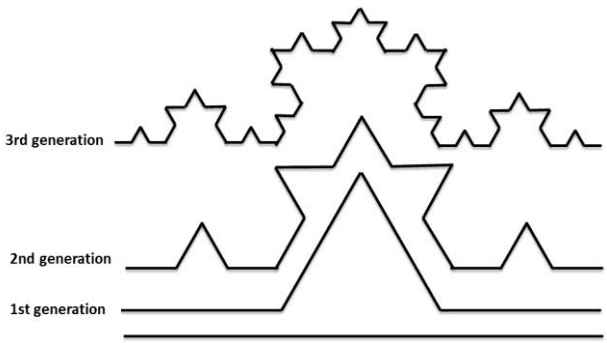


Figure 1. The Koch snowflake model

The algorithm of Von Koch snowflake [11] is as follows:

1. Draw in a line,
2. Divide the line into three equal parts,
3. Draw an equilateral triangle starting from the middle part,
4. Erase the middle part of the equilateral triangle,
5. Repeat steps 2, 3 and 4 in sequence.

Figure 1 is exponentially scaled as L length one part in the first generation, L/3 length 4 parts in the second generation, L/9 length 16 parts in the third generation, and then L/27 length 48 parts in the fourth generation. The fractal curve that occurs here has the property of self-similarity. If a certain part of the curve is selected and enlarged, it appears that the figure is the same as the original one. That is, the curve is independent of scale changes. The most important feature of this fractal curve is that it can be easily calculated without being bound to any formulas. The dimension of this model that emerges is different from the topological dimension of the classical Euclidean geometry. Fractal geometry has different features, even though it is an extension of the classical Euclidean geometry [1]. The concept of fractional dimension is mentioned when the dimensions of objects are defined. The lines forming the boundaries of fractal structures cannot be considered as one-dimensional lines because of roughness. Fractal figures are obtained through iteration, and thanks to this iteration the space remains limited as the edges go to infinity. For this reason, the concept of "fractal dimension" has emerged. The fractal dimension of an object gives information about the protruding, curved structure of

that object and the complexity of this structure. Fractal dimension is not a topological concept, it is a metric concept. The fractal dimension is indicated with *D*, also called as the fractional dimension or self-similarity dimension, it is identical to the Hausdorff dimension [12].
In Figure 1, it is seen that the circumference of this curve grows by 4/3 at every stage of the formation of the Koch curve. According to Mandelbrot's Koch snowflake construction [2, 13], the parameters of the Koch snowflake model is defined in Table 1.

Table 1. The parameters of the Koch snowflake model implementation [13].

Parameters	Defination
n	The number of iterations
N_n	The number of self-similar sides
L_n	The length of a single side
ℓ_n	The length of the perimeter
A_n	The snowflake's area
Δ	The area of the initial triangle

In Figure 1, the length of an initial side is 1 for $n = 0$. The fractal dimension is calculated for first generation of the Koch snowflake model illustrated in Figure 2 as follows [13].

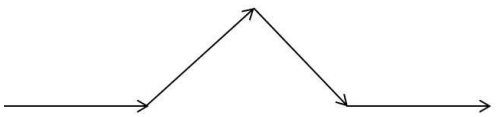


Figure 2. The first generation of the Koch snowflake model

$$N_n = 3x4^n \tag{1}$$
$$L_n = (1/3)^n \tag{2}$$
$$\ell_n = N_n L_n = 3\ (4/3)^n \tag{3}$$
$$A_n = A_{n-1} + \frac{1}{4} N_n L_n^2\ \Delta \tag{4}$$
$$= A_{n-1} + \frac{1}{3}\left(\frac{4}{9}\right)^{n-1}\ \Delta \tag{5}$$

$(A_0 = \Delta)$ yields,

$$A_n = \frac{1}{5}\left[8 - 3\ \left(\frac{4}{9}\right)^n\right]\ \Delta,\ n \to \infty, \tag{6}$$
$$A_\infty = \frac{8}{5}\ \Delta \tag{7}$$

The fractal dimension is then,

$$D = -\lim_{n\to\infty} \frac{\ln N_n}{\ln L_n} \tag{8}$$
$$= \log_3 4 \tag{9}$$
$$= \frac{2\ln 2}{\ln 3} \tag{10}$$
$$= 1.261859\ \ldots \tag{11}$$

These curves, including the reduced or rotated samples of themselves like Von Koch, Sierpinsky's [12], are deterministic fractals. This kind of fractals is produced by algorithms that are designed to be iterative or recursive depending on a specific rule. The structures that are used to simulate nature and which are obtained by methods that contain randomness in their structures are defined as random fractals. These methods can be transformed into computer programs using recursive algorithms. The best example that can be given to random fractals from nature is the tree model (Figure 3). The algorithm of branching tree model is as follows [11]:

1. Draw in a line,
2. At the end of the line, rotate to the left/right to angle δ and draw a line.
3. Repeat step 2 for the new branch.

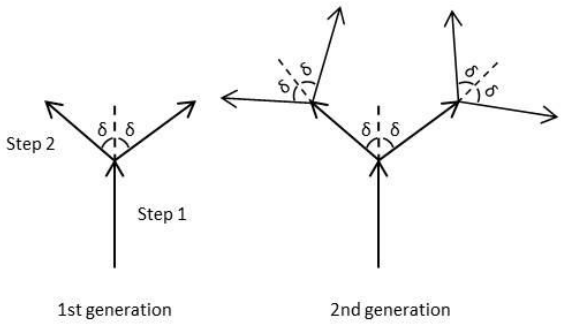


Figure 3. Fractal tree model

The rules defined with angles (δ) in the fractals are slightly different and difficult compared to the deterministic constructions. Each new branch that is created must be related to the previous branch angle. For this process, a transformation matrix is defined following the rotation.
Mitchell Feingenbaum, who studied mathematical fractals, demonstrated a close relationship between the fractals and chaos. The chaos in the nature can be described as the division into new branches and the development towards different directions of the "bifurcation" mechanism that allows the complex structure to emerge the system. There is no pattern in chaos, even though they are similar, nothing is the same, and each piece develops a whole jointly although they follow a separate path [14].The complicated structures in the nature can be explained by the Lindenmayer System with simpler rules.

3. Lindenmayer System

Aristide Lindenmayer developed a grammar-based system in 1968 in order to model the growth patterns of the plants. The Lindenmayer system provides a mechanism that can produce a small number of rules for fractal structures that require complicated and versatile rules. It is a system that helps modeling the structures that can be branched like plants. Along with the iterative algorithms and transformation matrices used in the fractal modeling, string/charset (sentence) appears in L-systems. It consists of 3 basic components [15, 16].

- **Alphabets:**An alphabet of the L-System is a set of (string) characters consisting of the current characters defined in it. For example: "ABC"
- **Axiom:**It is a character that identifies the initial state of the system, or it is a sentence (formed with characters in the alphabet). For example: "AAA" or "B" or "ACBAB."
- **Rule:**The rules defined in the L-system are applied via the axiom, and this is repeated to produce new sentences. A rule in the L-system contains two components; predecessor and successor. For example: In the rule defined as "A→AB", place where A is located is replaced by AB.

The application begins with the initial key word defined through the axiom, replacing each letter or symbol with another symbol or sentence using the rules defined in the L-system to produce a new sentence. Rules can be applied as much as desired. The sentence obtained this way is transformed into a graphical image using the turtle drawing algorithm [11, 15, 16].
An L-system is indicated with the function G = (V, ω, P). The alphabet of the system *V* is a set of points consisting of variable values. ω represents the starting point called the axiom. *P* indicates the set of production rules [16].
The product P is written as a → v, and it means writing v in each a condition. Thus instead of a symbol, the corresponding symbols take place. If no product is defined for a symbol, the symbol remains constant. That is, it becomes as a→a.
Lindenmayer's original L-system begins by modeling the growth of algae. In this model, the system is defined as follows:
Alphabets: A B
Axiom: A
Rules: (A→AB) (B→A)

Simulation starts with axiom and the generation number is 0. The number of generations can be gradually increased as illustrated in Figure 4 [11, 16].

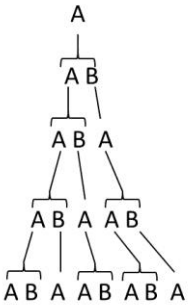


Figure 4. The modeling of the growth of algea for the fourth generation.

The turtle drawing algorithm is a tool for drawing the geometric description of the sentence created by rules. A state of the turtle is defined as (x, y, α). The Cartesian coordinates (x, y) indicates the location knowledge of the turtle. The angle α indicates the direction in which the turtle is facing [16].
Turtle graphic is created with the following some alphabets. With the turtle drawing algorithm, Koch curve can be simulated as follows:

F = draw a line with length d and move forward
f = move forward without drawing a line with length d
+= turn to the left with δ angle. The new state of the turtle is defined as (x,y,α + δ).
-= turn to the right with δ angle. The new state of the turtle is defined as (x,y,α - δ)

The turtle graph starts to be drawn with d length and δ angle. The function of the Koch curve is defined as follows:
$G(V, \omega, P) = \{(F, \delta, +, -) \in V, \omega: F, P: F \rightarrow F + F - - F + F\}, \delta = 60°$

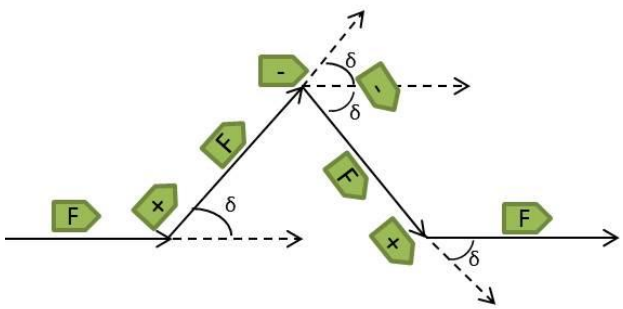


Figure 5. The turtle graph of the Koch snowflake for first generation

4. Fractal Vessel Structure Model

As noted above, fractal samples can be reproduced in different dimensions and patterns. The example of fractal geometry in plants can be demonstrated as the source of the vessel model in the cardiovascular system [17]. The branching structure of the plants is similar to the vascular structure covering the human body. For this reason, L-system algorithm is an important source for our agent based blood vessel model illustrated in Figure 6.
In Figure 6, each agent has its own identity knowledge (id), location knowledge (x, y, z), and radius knowledge (r) [3].We have just added the generation and angle parameters to these parameters to provide fractal branching depth. In our previous study, the connector agents have provided the branching in the bifurcation vessel model.

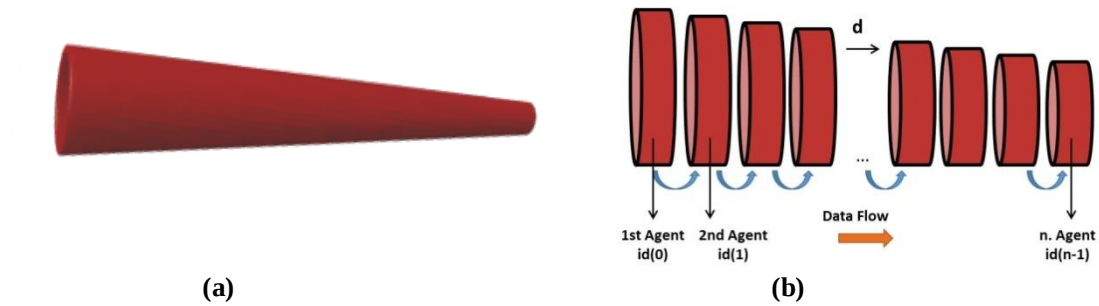


Figure 6. a) Vessel model, **b)** Agent based vessel model

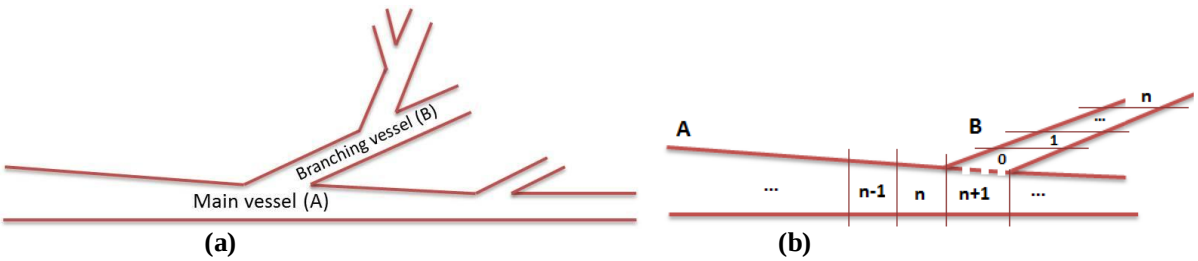


Figure 7. a) Main vessel (A) model and branching vessel (B) model, **b)** n is the number of agents. B_0 is the connector agent [3].

L-system providesa better structure for the development of the branching vessel model through a process of string re-writing to our agent based model. L-system is extended with new constants, '[' and ']' for delimitation of the branch.We have simulated agent based branching vessel model according to L-system production rules. The function of this model is defined as follows:

$G(V, \omega, P) = \{(F, \delta, +, [])\in V, \omega: F, P: F\rightarrow F[+F]F \}$

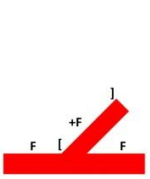
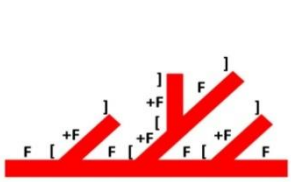
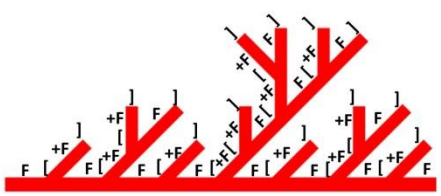
$\delta = 60$

L-system production rules are interpreted by the turtle [7, 11, 16] as follows:

F : draw a line with length d and move forward
+ : turn to the left with δ angle. The new state of the turtle is defined as $(x,y,\alpha + \delta)$.
[: save current location of the turtle
] : restore previous location of the turtle.

We have defined L-system production rules and obtained turtle graphic for two generations illustrated in Table 2.

Table 2. Turtle graphic of L-system for two generations

L-system	Turtle graphic		
	Generation 0	**Generation 1**	**Generation 2**
Alphabets: F Constants: + [ ] Axiom: F Rules: F→F[+F]F δ: 45°	F[+F]F	F[+F]F[+F[+F]F]F[+F]F	F[+F]F[+F[+F]F]F[+F]F[+F[+F]F]F[+F[+F]F]F [+F]F[+F]F[+F[+F]F]F[+F]F
			

We have simulated the turtle drawing algorithm in the Repast Simphony [18] simulation environment illustrated in Figure 8. According to L-system production rules, we assume that the length and the radius of the agents remain constant**.** Also, we have simulated the agents in 2 dimension (2D) Cartesian coordinates.

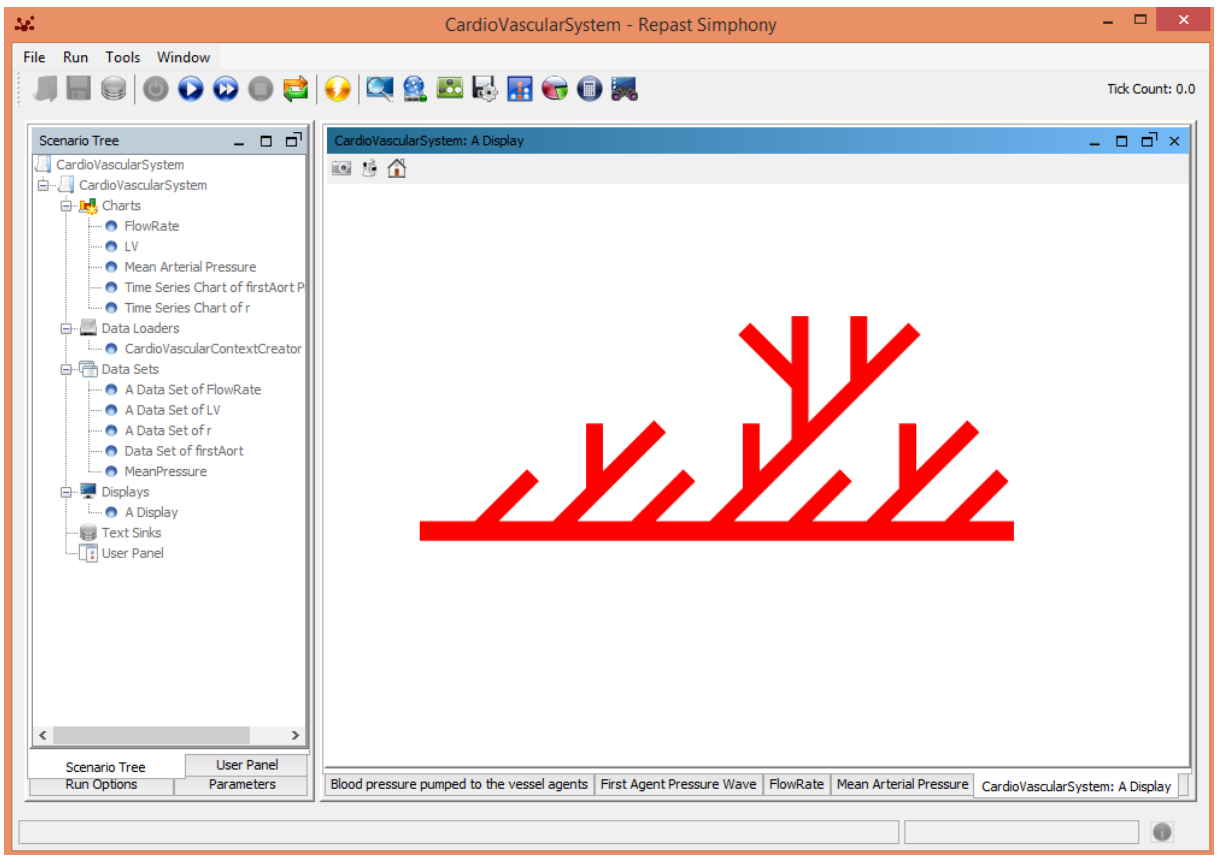


Figure 8. The simulation of turtle graphic algorithm

5. Conclusions

In this study, we have observed that the string rewriting technique of L-systems helps the blood vessel model which have developed in our previous study to add new branching structures. Our abstraction approach offers the basic idea to define

complex structures using L-systems. L-systems are developed by successively generating recursive functions using a set of rewriting rules. However, we have not found any study which includes applications of L-systems in the agent based modeling technique in the literature. Therefore, we have just simulated a fractal vessel structure according to the L-systems algorithm. Following this study, we aim to observe some dynamic physiological parameters (bloodflow, bloodpressure, velocity, vesselresistance, etc.) of a vessel network defined by agents' behaviors.

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