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Existence and Multiplicity of Solutions for Polyharmonic Elliptic Problem on Eistein Manifold Involving GJMS Operator

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Abstract

Let (M^n, g) be a compact Eistein Riemannian manifold of dimension n and denote by S_g the scalar curvature of g. Let k be a positive integer. We prove that $n > 2k$ and under some geometric conditions that the elliptic equation:

$$P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\,u + \lambda h(x)\,|u|^{q-2}\,u$$

has two distinct solutions: one positive and other negative.

Keywords

1. Introduction.

In 1992, Graham-Jenne-Mason-Sparling in [8], discovered a particularly conformally fourth order operator de.ned on 2k-dimensional smooth Riemannian manifolds. This operator named in short GJMS operator. Given (M^n, g) a smooth, compact Riemannian manifold of dimension $n \geq 3$. Let k be a positive integer such that $2k < n$ and denote by S_g and Ric_g the scalar and Ricci curvature of g. Let $u \in C^{2k}(M)$, the GJMS has a principle part $(\Delta_g)^k u,\ k \in \mathbb{N}$. More precisely, for any riemannian metric g on M, the operator P_g^k is given by:

$$P_g^k : C^\infty(M) \to C^\infty(M)$$

such that for $u \in C^\infty(M)$

$$P_g^k(u) := (\Delta_g)^k u + lot$$

Where $\Delta_g u$: is the Laplace-Beltrami operator.

This operator is conformally invariant in the sense that if $\tilde{g} := \varphi^{\frac{4}{n-2k}} g$, for all $\varphi \in C^\infty(M), \varphi > 0$:

$$P_g^k(\varphi u) = \varphi^{\frac{n+2k}{n-2k}} . P_{\tilde{g}}^k(u).$$

Moreover, P_g^k is self-adjoint with respect to L^2-scalar product. A scalar invariant is associated to this operator, namely the Q-curvature, denoted by Q_g^n. In the specific case $2k < n$, this geometric quantity has been obtained when we take $\varphi \equiv 1$ as

$$Q_g^k := \frac{2}{n-2k} P_g^k(1).$$

Observe that when (M^n, g) is Eistein, P_g^k is of constant coefficients, explicitly:

$$P_g^k(u) := \prod_{i=1}^{i=k} \left(\Delta_g - \frac{(n+2i-2).(n-2i) S_g}{4n(n-1)} \right) u$$

The GJMS operator enjoys an interesting conformal properties that are very simular to ones of conformal Laplacian and Paneitz-Branson operators. Recently, some remarkable developments have been achieved in the existence results of prescribed Q-curvature problem on Eisteinian manifolds, we refer the reader to [5], [6], [9] [11] and [12].

We denote by $H_k^2(M)$ is the standart Sobolev space, where for $1 \leq m \leq k$, $H_m^2(M)$ which the completion of $C^\infty(M)$ with respect to the norm:

$$\|u\|^2_{H_m^2(M)} := \sum_{l=0}^{l=m} \int_M \left| \Delta_g^{\frac{l}{2}} u \right|^2 d\mu(g).$$

where

$$\Delta_g^{\frac{m}{2}} u := \begin{cases} \Delta_g^l u & \text{if } m = 2l \quad \text{is even} \\ \nabla_g \Delta_g^m u & \text{if } m = 2l+1 \quad \text{is odd} \end{cases}$$

We denote by $K(n, k) > 0$ the best constant in the Sobolev's continuous embedding $D^{k,2}(\mathbb{R}^n) \hookrightarrow L^{2_k^\sharp}(\mathbb{R}^n)$

$$K(n,k) := \inf_{u \in D^{k,2}(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} \left(\Delta^{\frac{k}{2}} u \right)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} dx \right)^{\frac{2}{2_k^\sharp}}} \tag{1}$$

Thanks to the series of results of Lions [17] and Swanson [19] that the extremal functions for the Sobolev inequality [18] exist and are exactly multiples of the functions:

$$u_\alpha(x) := \alpha_{n,k}^{\frac{n-2k}{4k}} \left(\frac{\lambda}{1+\lambda^2 |x-a|^2} \right)^{\frac{n}{2_k^\sharp}}$$

Where $a \in \mathbb{R}^n$, $\lambda > 0$,

$$\alpha_{n,k} = \Pi_{l=-k}^{l=k-1} (n+2l).$$

Now, we use the following Sobolev inequalities proven in to [18].

Theorem 1 *Let (M^n, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$: For any $\epsilon > 0$, there is a constant $A_\epsilon \in \mathbb{R}$ such that*

$$\forall u \in H_k^2(M) : \left(\int_M |u|^{2_k^\sharp} d\mu(g) \right)^{\frac{2}{2_k^\sharp}} \leq (1+\epsilon) K(n,k) \int_M \left| \Delta_g^{\frac{m}{2}} u \right|^2 d\mu(g) + A_\epsilon \|u\|^2_{H_{k-1}^2(M)}$$

Concerning the polyharmonic case, Pucci and Serrin in [8] have studied the following elliptic problem:

$$\begin{cases} (-\Delta)^k u = |u|^{2_k^\sharp - 2} u + \lambda u & \text{in} \quad \Omega, \\ \left(\frac{\partial}{\partial \nu} \right)^j u \Big|_{\partial\Omega} = 0, j = 0, 1, 2.... (k-1). \end{cases}$$

when Ω is the unit ball of $\mathbb{R}^n$ and they proved that it admits nontrivial radial symmetric solution for all n $\geq$ 8 that if $\lambda \in (0, \lambda_1)$.

In [4], Benalili and Bekiri have established the existence of nodal solutions of the following polyharmonic elliptic problem on a Riemannian manifold with boundary:

$$\begin{cases} P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\,u & \text{in} \quad M, \\ \left(\frac{\partial}{\partial \nu}\right)^j u\Big|_{\partial M} = \varphi_j, \ j = 0, 1, 2.... (k-1). \end{cases}$$

where $\varphi_1, \cdots, \varphi_{k-1}$ are smooth functions in ∂M.

Recently in [22], the author has been intrested by the existence of multiple solutions of the *Q*-curvature problem with perturbation of the form:

$$P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\,u + h \quad \text{in} \quad M,$$

Where $f \in C^\infty(M)$ with $f > 0$ and *h* belongs to a suitable Sobolev space.

Recently in [24], the author has shown that the following nonhomogenous *Q*-curvature problem with perturbation has at least two solutions of the form:

$$P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\,u + h \quad \text{in} \quad M,$$

Where $f \in C^\infty(M)$ with $f > 0$, $h \neq 0$ and $\|h\|_{(H_k^2(M))^*}$ is small enough and for every $k \geq 3$.

In this paper, We are concerned with the following problem. Let (M^n, *g*) be a smooth, compact Eistein Riemannian manifold of dimension $n \geq 3$ and denote by S_g and Ric_g the scalar and Ricci curvature of *g*. We investigate the existence and multiplicity of solutions for polyharmonic operators with critical exponent:

$$P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\,u + \lambda h(x)\,|u|^{q-2}\,u.$$

Where $h \in L^d(M)$, with $d > \frac{2_k^\sharp}{2_k^\sharp - q} := \alpha, \ 2 < q < N$ and $\lambda > 0$ a real parameter.

Throughout this section, we consider the energy functional J_λ, for each $u \in H_k^2(M)$,

$$J_\lambda(u) = \frac{1}{2}\int_M P_g^k(u)u.d\mu(g) - \frac{\lambda}{q}\int_M h(x)\,|u|^q\,d\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x)\,|u|^{2_k^\sharp}\,d\mu(g)$$

Here, we assume that:

(*Hyp1*) The operator $P_g^k : C^\infty(M) \longrightarrow C^\infty(M)$ is coercive i.e

$$\exists \Lambda > 0 : \left\langle P_g^k(u); u \right\rangle \geq \Lambda \|u\|_{H_k^2(M)}^2, \forall u \in H_k^2(M).$$

(*Hyp2*) The function $f : M \longrightarrow \mathbb{R}$ is positive C^∞-function on *M*.

(*Hyp3*) If the function *h* doesn't vanishe almost everywhere on *M*.

(*Hyp4*) The parameter $0 < \lambda < \lambda_1$ such that

$$\lambda_1 := \frac{q\left(2_k^\sharp - 2\right)}{2\left(2_k^\sharp - q\right)} \Lambda^{\frac{q}{2}} \left[\max\left((1+\epsilon)K(n,k), A_\epsilon\right)\right]^{-\frac{q}{2}} \|h\|_\alpha^{-1}$$

2. Some Preparatory Lemmas

Before stating the proofs of our main results, we give some fruitful lemmas that help us in the following proofs:

Lemma 2 *Assume that* $P_g^k : C^\infty(M) \to C^\infty(M)$ *is coercive. Then* $|u|_{P_g^k} := \left[\int_M P_g^k(u).u.d\mu(g)\right]^{\frac{1}{2}}$ *is a norm on* $H_k^2(M)$ *equivalent to* $\|u\|_{H_k^2(M)}$.

Proof. (See the proof in [19]).

Now, we first give an example of manifolds where the GJMS operator P_g^k is coercive and it will be our working manifold in the whole following analysis.

Lemma 3 *Let (M^n, g) be a smooth Eistein Riemannian manifold of dimension* $n \geq 3$ *and denote by* S_g *its scalar curvature. Let k be a positive integer such that* $2k < n$. *Then,* P_g^k *is coercive if only if* $S_g > 0$

Our working norm as follow, for all $u \in H_k^2(M)$:

$$|u|_{P_g^k} = \left[\int_M P_g^k(u)u.d\mu(g)\right]^{\frac{1}{2}}$$

In this setion the main tool is the Mountain-Pass lemma of Ambrossetti-Rabinowitz :

Lemma 4 *let* $J \in C^1(E,\mathbb{R})$ *where* $(E,\|.\|_E)$ *is a Banach space. We assume that:*

(i) $J(0) = 0$.

(ii) $\exists r, R > 0$ *such that* $J(u) \geq R > 0$ *for all* $u \in E$ *such that* $\|u\|_E = r$.

(iii) $\exists v \in E$ *such that* $lim\ sup_{t\longrightarrow+\infty} J(tv) < 0$.

Let

$$c = \min_{\eta\in\Gamma} \max_{t\in[0,1]} \left(J\left(\eta(t)\right)\right)$$

where

$$\Gamma = \left\{\eta \in C^1\left([0;1];\ E\right) :\ \eta(0) = 0,\ \eta(1) = v\right\}$$

Then there exists a sequene $(u_n)_n$ *in E such that :*

$$J(u_n) \longrightarrow c \quad and \quad \nabla J(u_n) \longrightarrow 0 \ in\ E^*$$

Moreover, we have that

$$c \leq \sup_{t\geq 0} J(tv)$$

We see that J_λ is a C^1 functional and its Fréchet derivative is given by:

$$\langle \nabla J_\lambda(u), v\rangle =$$

$$\int_M P_g^k(u).vd\mu(g) - \lambda\int_M h(x)|u|^{q-2}uvd\mu(g) - \int_M f(x)|u|^{2_k^\sharp-2}uvd\mu(g).$$

We shall prove that the functional J_λ verifies the Mountain-Pass geometry conditions, namely.

Lemma 5 *Suppose either or that. Then* J_λ *verifies the following conditions:*

1. There exist constants $r, R > 0$, *such that* $J_\lambda(u) \geq R > 0$, $|u|_{P_g^k} = r$.

2. There exists $v \in H_k^2(M)$ *with* $|v|_{P_g^k} > r$, *such that* $J_\lambda(v) < 0$.

Proof.

1. We want to prove this lemma in two parts:

Let $u \in H_k^2(M)$ such that $|u|_{P_g^k} = r > 0$; then we have

$$J_\lambda(u) = \frac{1}{2}|u|^2_{P_g^k} - \frac{\lambda}{q}\int_M h(x)\,|u|^q\,d\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x)\,|u|^{2_k^\sharp}\,d\mu(g)$$

By Hölder and Sobolev inequalities, we deduce that

$$J_\lambda(u) \geq \frac{1}{2}|u|^2_{P_g^k} - \frac{\lambda}{q}(\max((1+\epsilon)K(n,k),A_\epsilon))^{\frac{q}{2}}\,\|h\|_\alpha\,\|u\|^q_{H_k^2(M)}$$

$$-\frac{1}{2_k^\sharp}(\max((1+\epsilon)K(n,k),A_\epsilon))^{\frac{2_k^\sharp}{2}}\max_{x\in M}(f(x))\,\|u\|^{2_k^\sharp}_{H_k^2(M)}$$

Hence if (Hyp1) is satisfied then, there exists a constant $\Lambda > 0$, such that :

$$J_\lambda(u) \geq \frac{1}{2}|u|^2_{P_g^k} - \frac{\lambda}{q}\Lambda^{-\frac{q}{2}}(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{\frac{q}{2}}\,\|h\|^q_\alpha\,|u|^q_{P_g^k}$$

$$-\frac{1}{2_k^\sharp}\Lambda^{-\frac{N}{2}}(\max((1+\varepsilon)K_\circ,A_\varepsilon))^{\frac{2_k^\sharp}{2}}\max_{x\in M}(f(x))\,|u|^{2_k^\sharp}_{P_g^k}$$

With $|u|_{P_g^k} = r$, we get

$$J_\lambda(u) \geq r^2 \times \left(\frac{1}{2} - \lambda\frac{\Lambda^{-\frac{q}{2}}}{q}(\max((1+\epsilon)K(n,k),A_\epsilon))^{\frac{q}{2}}\,\|h\|_\alpha\,r^{q-2}\right.$$

$$\left.-\frac{\Lambda^{-\frac{2_k^\sharp}{2}}}{2_k^\sharp}(\max((1+\epsilon)K(n,k),A_\epsilon))^{\frac{2_k^\sharp}{2}}\max_{x\in M}(f(x))r^{2_k^\sharp-2}\right)$$

Then there are $r, R > 0$, such that for all $u \in H_k^2(M)$ with $|u|_{P_g^k} = r$ we get $J_\lambda(u) \geq R > 0$.

2. Let $t > 0$ and $u \in H_k^2(M) - \{0\}$; thus

$$J_\lambda(tu) = \frac{t^2}{2}|u|^2_{P_g^k} - \frac{\lambda t^q}{q}\int_M h(x)\,|u|^q\,d\mu(g) - \frac{t^{2_k^\sharp}}{2_k^\sharp}\int_M f(x)\,|u|^{2_k^\sharp}\,d\mu(g)$$

Since $2 < q < 2_k^\sharp$, we have that

$$J_\lambda(tu) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty$$

This proves lemma.

Applying the Mountain-Pass lemma, there exists a sequence $(u_n)_n$ in $H_k^2(M)$ such that:

$$J(u_n) \longrightarrow c_\lambda \quad \text{and} \quad \nabla J(u_n) \longrightarrow 0 \quad \text{in} \quad \left(H_k^2(M)\right)^*$$

Where

$$\Gamma = \left\{\eta \in C^1\left([0;1];\ H_k^2(M)\right) :\ \eta(0) = 0,\ \eta(1) = v\right\}$$

and

$$c_\lambda = \min_{\eta\in\Gamma}\max_{t\in[0,1]}\left(J_\lambda(\eta(t))\right)$$

Let

$$\lambda_1 = \frac{q\left(2_k^\sharp - 2\right)}{2\left(2_k^\sharp - q\right)} \Lambda^{\frac{q}{2}} \left(\max\left((1+\epsilon)K(n,k), A_\epsilon\right)\right)^{-\frac{q}{2}} \|h\|_\alpha^{-1}$$

Then, we have the following lemmas.

Lemma 6 *Let (M, g) is an n-dimensional compact, smooth and orientable Riemannian manifold with* $n \geq 5$ *with we suppose the conditions* (h^1), (h^2) *and* (h^3) *are satisfying.*

Then each Palais-Smale sequence at level c_λ *is bounded in* $H_2^2(M)$.

Proof.

Let $(u_m)_m$ a sequence in $H_k^2(M)$ such that :

$$J(u_m) \longrightarrow c_\lambda \quad \text{and} \quad \nabla J(u_m) \longrightarrow 0 \quad \text{in} \quad \left(H_k^2(M)\right)^*$$

Then we have:

$$J(u_m) - \frac{1}{2_k^\sharp} \langle \nabla J(u_m), u_m \rangle = \left(\frac{1}{2} - \frac{1}{2_k^\sharp}\right) \|u_m\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{2_k^\sharp}\right) \int_M h(x) |u_m|^q \, dv(g)$$

By Hölder and Sobolev inequalities, we get that

$$J(u_m) - \frac{1}{2_k^\sharp} \langle \nabla J(u_m), u_m \rangle = c_\lambda + o(1)$$

$$\geq \left(\frac{1}{2} - \frac{1}{2_k^\sharp}\right) \|u_m\|^2 - \left(\frac{1}{q} - \frac{1}{2_k^\sharp}\right) \left(\max\left((1+\epsilon)K(n,k), A_\epsilon\right)\right)^{\frac{q}{2}} \|h\|_\alpha \|u_m\|_{H_k^2(M)}^q$$

By the coerciveness of the operator P_g^k , there exists a constant $\Lambda > 0$, such that :

$$c_\lambda + o(1) \geq \left(\frac{1}{2} - \frac{1}{2_k^\sharp}\right) \|u_m\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{2_k^\sharp}\right) \Lambda^{-\frac{q}{2}} \left(\max\left((1+\epsilon)K(n,k), A_\epsilon\right)\right)^{\frac{q}{2}} \|h\|_\alpha \|u_m\|^q$$

we consider the following when $\|u_n\| \geq 1$, then we have

$$c_\lambda + o(1) \geq \left(\frac{2_k^\sharp - 2}{2} - \lambda \frac{2_k^\sharp - q}{q} \Lambda^{-\frac{q}{2}} \left(\max\left((1+\epsilon)K(n,k), A_\epsilon\right)\right)^{\frac{q}{2}} \|h\|_\alpha\right) \frac{\|u_m\|^q}{2_k^\sharp}$$

Hence if (*Hyp*2) is satisfied then we have

$$\|u_m\| \leq \left(\frac{c_\lambda}{\frac{2_k^\sharp - 2}{2 2_k^\sharp} - \lambda \frac{2_k^\sharp - q}{q 2_k^\sharp} \Lambda^{-\frac{q}{2}} \left(\max\left((1+\epsilon)K(n,k), A_\epsilon\right)\right)^{\frac{q}{2}} \|h\|_\alpha}\right)^{\frac{1}{q}} + o(1)$$

Thus, the sequence $(u_m)_m$ is bounded in $H_k^2(M)$.

Theorem 7 *Let* (M^n, *g) is an n-dimensional compact, smooth and orientable Riemannian manifold with* $n \geq 5$. *Let be* $(u_m)_m$ *a Palais-Smale sequence at level* c_λ *and we suppose the conditions (Hyp1), (Hyp2) and (Hyp3) are satisfying.*

Then there is a subsequence of $(u_m)_m$ *converging strongly in* $H_k^2(M)$.

Proof.

Let $(u_m)_m$ be a $(P.S)_{c_\lambda}$ sequence. By using Lemma and from the reflexivity of $H_2^2(M)$ and the compact imbedding

theorem, we conclude that there exists a subsequence $(u_m)_m$ and $u \in H_2^2(M)$ such that:

(1). $u_m \longrightarrow u$ weakly in $H_2^2(M)$.

(2). $u_m \longrightarrow u$ strongly in $L^p(M)$ for $1 < p < N = \frac{2n}{n-4}$.

(3). $\nabla u_m \to \nabla u$ strongly in $L^q(M)$ for $1 < q < 2^* = \frac{2n}{n-2}$.

(4). $u_m \longrightarrow u$ a.e in *M*.

Then we deduce that

$$\left| \int_M a(x) \left| \nabla (u_m - u) \right|^2 dv(g) \right| \leq \|a\|_r \left\| \nabla (u_m - u) \right\|^2_{\frac{2r}{r-1}} \to 0$$ strongly because $\frac{2r}{r-1} < 2^* = \frac{2n}{n-2}$

and

$$\left| \int_M b(x) \left| u_m - u \right|^2 dv(g) \right| \leq \|b\|_s \left\| u_m - u \right\|^2_{\frac{2s}{s-1}} \to 0$$ strongly since $\frac{2s}{s-1} < N = \frac{2n}{n-4}$

and also

$$\left| \int_M h(x) \left| u_m - u \right|^q dv(g) \right| \leq \|b\|_d \left\| u_m - u \right\|^q_{\frac{dq}{d-1}} \to 0$$ strongly since $\frac{dq}{d-1} < N = \frac{2n}{n-4}$

After these preliminaries we can prove that $v_m := u_m - u$ converges to 0 strongly in $H_2^2(M)$.

Using Br ézis-Lieb Lemma, we obtain

$$\|\Delta_g u_m\|_2^2 - \|\Delta_g u\|_2^2 = \|\Delta_g v_m\|_2^2 + o(1)$$

and

$$\int_M f(x) \left(|u_m|^N - |u|^N \right) dv(g) = \int_M f(x) |v_m|^N dv(g) + o(1)$$

and also

$$\int_M h(x) \left(|u_m|^q - |u|^q \right) dv(g) = \int_M h(x) |v_m|^N dv(g) + o(1)$$

Taking account of $v_m \to 0$ weakly in $H_2^2(M)$, we have for all $\phi \in L^2(M)$

$$\int_M v_m \Delta_g^2 \phi dv(g) = o(1)$$

Then,

$$\int_M u \Delta_g^2 \phi dv(g) + o(1) = \int_M \Delta_g \phi \Delta_g u dv(g) + o(1)$$

Since $u_m \longrightarrow u$ weakly in $H_2^2(M)$ i.e. for all $\phi \in (H_2^2(M))^*$, we have

$$\phi(v_m) = \phi\left(u_m - u^+\right) = o(1)$$

we take in particular $\phi = P_g^k(\varphi)$ for $\varphi \in H_2^2(M)$:

$$\int_M P_g(\varphi) v_m dv(g) = o(1)$$

If we take $\varphi = u$, we obtain

$$\int_M (\Delta_g u_m \Delta_g u - a(x) g \langle \nabla u_m, \nabla u \rangle + b(x) u_m u)\, dv(g) = \|u\|^2 + o(1)$$

and also if we put $\varphi = u_m$,

$$\int_M (\Delta_g u_m \Delta_g u - a(x) g \langle \nabla u_m, \nabla u \rangle + b(x) u_m u)\, dv(g) = \|u_m\|^2 + o(1)$$

and also

$$\int_M \left(\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u\right) dv(g) = \|u\|^2 + o(1)$$

And by letting $m \to +\infty$

$$\int_M \left(\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u\right) dv(g) \to \int_M \left(\lambda |u|^q + f(x) |u|^N\right) dv(g)$$

Taking account of we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2} \|\Delta_g (u_m - u)\|_2^2 - \frac{1}{N} \int_M f(x) |u_m - u|^N\, dv(g) + o(1)$$

And taking account of $u_m - u \to 0$ weakly in $H_2^2(M)$, we test by $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$

$$\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u \rangle = o(1)$$

$$= \|\Delta_g (u_m - u)\|_2^2 - \int_M f(x) |u_m - u|^N\, dv(g) = o(1)$$

So

$$\|\Delta_g (u_m - u)\|_2^2 = \int_M f(x) |u_m - u|^N\, dv(g) + o(1)$$

We take account of (??), we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2} \|\Delta_g (u_m - u)\|_2^2 - \frac{1}{N} \|\Delta_g (u_m - u)\|_2^2 + o(1)$$

i.e.

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{n} \|\Delta_g (u_m - u)\|_2^2$$

By Sobolev inequality, we have for all $u \in H_2^2(M)$

$$\|u\|_N^2 \leq (1 + \varepsilon) K_\circ \int_M (\Delta_g u)^2 + |\nabla_g u|^2\, dv(g) + A_\varepsilon \int_M u^2 dv(g)$$

We test the Sobolev inequality by $u_m \to u$, we get

$$\|u_m - u\|_N^2 \leq (1 + \varepsilon) K_\circ \int_M (\Delta_g (u_m - u))^2\, dv(g) + o(1)$$

So

$$\int_M f(x)\left|u_m-u\right|^N dv(g) \leq \max_{x\in M} f(x) \int_M \left|u_m-u\right|^N dv(g)$$

Hence taking account of (2), we obtain

$$\int_M f(x)\left|u_m-u\right|^N dv(g) \leq (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m-u)\right\|_2^N + o(1)$$

Taking account of equality (2), one writes

$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m-u)\right\|_2^N + o(1)$$

Then

$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g\left(u_m-u\right)\right\|_2^N + o(1)$$

Taking account that $\|.\|$ and $\|.\|_{H_2^2(M)}$ are equivalent, there exists a constant $\Lambda>0$ such that

$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g\left(u_m-u\right)\right\|_2^N + o(1)$$

$$o(1) \geq \left\|\Delta_g\left(u_m-u\right)\right\|_2^2 \left(1-(1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g\left(u_m-u\right)\right\|_2^{N-2}\right) + o(1)$$

Hence if

$$\limsup_{m\to+\infty} \left\|\Delta_g\left(u_m-u\right)\right\|_2^{N-2} < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}} K_\circ^{\frac{n}{n-4}} \max_{x\in M} f(x)}$$

We get

$$\frac{2}{n}\int_M \left(\Delta_g(u_m-u)\right)^2 dv(g) < c.$$

Since

$$c < \frac{2}{n\ K_\circ^{\frac{n}{4}} \left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}}$$

Then

$$\int_M \left(\Delta_g(u_m-u)\right)^2 dv(g) < \frac{1}{K_\circ^{\frac{n}{4}} \left(\max_{x\in M} f(x)\right)^{\frac{n-4}{4}}}.$$

Hence

$$o(1) \geq \left\|\Delta_g(u_m-u)\right\|_2^2 \underbrace{\left(1-(1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_\circ^{\frac{n}{n-4}} \left\|\Delta_g(u_m-u)\right\|_2^{N-2}\right)}_{>0} + o(1)$$

Or also

$$\left\|\Delta_g(u_m-u)\right\|_2^2 = o(1)$$

i.e. $u_m \to u$ converges strongly in $H_2^2(M)$.

3. Positivity and Multiplicity of Solutions

We consider the following two functionals: for each $u \in H_k^2(M)$:

$$J_\lambda^+(u) = \frac{1}{2}\int_M P_g^k(u)u.d\mu(g) - \frac{\lambda}{q}\int_M h(x)\left(u^+\right)^q d\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x)\left(u^+\right)^{2_k^\sharp} d\mu(g)$$

And

$$J_\lambda^-(u) = \frac{1}{2}\int_M P_g^k(u)u.d\mu(g) - \frac{\lambda}{q}\int_M h(x)\left(u^-\right)^q d\mu(g) - \frac{1}{2_k^\sharp}\int_M f(x)\left(u^-\right)^{2_k^\sharp} d\mu(g)$$

where

$$u^+ := \max(u, 0) \text{ and } u^- := \min(u, 0)$$

Then, critical points of J_λ^+ and J_λ^- are solutions to these equations respectively:

$$P_g^k(u) = f(x)u^{2_k^\sharp - 1} + \lambda h(x)u^{q-1} \tag{2}$$

and

$$P_g^k(u) = \left(f(x)\left|u^-\right|^{2_k^\sharp - 2} + \lambda h(x)\left|u^-\right|^{q-2}\right)u^-. \tag{3}$$

By applying the maximum principle *k*-times for the two previous equations and since $S_g > 0$; we obtain u^+ and u^- are solutions of the equations [13] and [14] respectively.

4. Test Functions

The purpose of this section is to prove that the geometric conditions obtained in [1] are verified. Let $(y^1, .., y^n)$ be normal coordinates centered at the point y_o where the function *f* attains its maximum. Let $B(y_o, \delta)$ be the ball centered at y_o and of radius δ with $0 < 2\delta < d$ (*d* the injectivity radius). Denote by $d\sigma$ the volume element of the (*n* - 1)-dimentional unit sphere S^{n-1} (1). We let also η be a smooth function equals to 1 on $B(y_o, \delta)$ and equals to 0 on $M - B(y_o, 2\delta)$.

Put

$$u_\epsilon(x) = \xi.\left(\frac{\epsilon}{r^2 + \epsilon^2}\right)^{\frac{n-2k}{2}}\eta(r)$$

where

$$\xi := \left[\left(\frac{\Pi_{-k\le j\le k-1}(n+2j)}{f(a)}\right]^{\frac{n-2k}{4k}}\right.$$

We know from the work of [20], that

$$\frac{1}{K(n,k)} = \Pi_{-k\le j\le k-1}(n+2j)\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right)^{\frac{2k}{n}}\pi^k$$

where Γ denotes the Gamma function.

We compute now these quantities:

$$\zeta =: \int_M f(x) \left|u_\epsilon(x)\right|^{2_k^\sharp} d\mu\left(g\right)$$

and

$$\gamma := \int_M P_g^k\left(u_\epsilon\right).u_\epsilon d\mu\left(g\right)$$

As in [6], we get

$$\int_M f(x) \left|u_\epsilon(x)\right|^{2_k^\sharp} d\mu\left(g\right) = \frac{1}{\left[K\left(n,k\right)\right]^{\frac{n}{2k}} \left[f(y_o)\right]^{\frac{n-2k}{2k}}} \times$$

$$\left(1 - \frac{3\Delta f(y_o) + f(y_o) S_g(y_o)}{6\left(n-2\right) f(y_o)} \epsilon^2 + o(\epsilon^2)\right).$$

and

$$\int_M P_g^k\left(u_\epsilon\right).u_\epsilon d\mu\left(g\right) = \frac{1}{\left[K\left(n,k\right)\right]^{\frac{n}{2k}} \left[f(y_o)\right]^{\frac{n-2k}{2k}}} \times$$

$$\left(1 - (\frac{\left(n^2+4n-20\right) S_g(y_o) + 24(n-1)a(y_o)}{6(n^2-4)(n-6)})\epsilon^2 + o(\epsilon^2)\right).$$

We consider the path

$$\gamma_\epsilon\left(t\right) := \frac{u_\epsilon}{\left\|u_\epsilon\right\|_{2_k^\sharp}} t$$

It suffices to show that there exists $\epsilon > 0$ such that

$$\sup_{t\geq 0} J_\lambda\left(\gamma_\epsilon\left(t\right)\right) < \frac{k}{n\left[K\left(n,k\right)\right]^{\frac{n}{2k}} \left[f(y_o)\right]^{\frac{n-2k}{2k}}}$$

where

$$J_\lambda\left(\gamma_\epsilon\left(t\right)\right) = \frac{t^2}{2\left\|u_\epsilon\right\|_{2_k^\sharp}^2} \int_M P_g^k(u_\epsilon).u_\epsilon d\mu(g) -$$

$$\frac{\lambda t^q}{q\left\|u_\epsilon\right\|_{2_k^\sharp}^q} \int_M h(x)\left|u_\epsilon\right|^q d\mu(g) - \frac{t^{2_k^\sharp}}{2_k^\sharp \left\|u_\epsilon\right\|_{2_k^\sharp}^{2_k^\sharp}} \int_M f(x)\left|u_\epsilon\right|^{2_k^\sharp} d\mu(g)$$

$$J_\lambda\left(\gamma_\epsilon\left(t\right)\right) = \frac{t^2}{2} A_\epsilon - \frac{\lambda t^q}{q} B_\epsilon - \frac{t^{2_k^\sharp}}{2_k^\sharp} C_\epsilon$$

Using the estimates above and we assume that

$$\frac{\Delta f(y_o)}{f(y_o)} < \frac{2\left(k+4\right)n^2 - 8\left(k+1\right)n - 24k}{3\left(n-2k\right)\left(n+2\right)(n-6)} S_g(y_o)$$

Then we get that

$$\sup_{t\geq 0} J_\lambda\left(\gamma_\epsilon\left(t\right)\right) < \frac{k}{n\left[K\left(n,k\right)\right]^{\frac{n}{2k}}\left[f\left(y_o\right)\right]^{\frac{n-2k}{2k}}}$$

Which achieves the proof completely.

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