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Discontinuous Galerkin Method of the Second Order Parabolic Differential Equation

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Abstract

This paper is concerned with the mathematical study of the discontinuous Galerkin finite element method for the parabolic differential equation. Finite element methods have been recognized as valuable in the numerical approximation of solutions to self-adjoint or nearly self-adjoint parabolic partial differential equation problems. The discontinuous Galerkin method is a vital numerical method with much mass compensation and more flexible meshing than other methods. These methods belong to a class of numerical methods for solving partial differential equations. They are based on weak formulations and with finite dimensional piecewise polynomial solution space and test function space. This study is focused on the discontinuous Galerkin method of the second order parabolic problem. The parabolic problem satisfies the condition of the existence and uniqueness of DG solution. The error analysis of this problem is also established. The main goal of this study is to explore the convergence of the solution of the DG method and show the validity of the results.

Keywords

Jump and average; variational formulation; discontinuous Galerkin; parabolic equation; error analysis

1. Introduction

This paper gives a theoretical concept to approximate the error of the solutions of a second order parabolic differential equation. The error analysis is usually based on the variational interpretation of the FEM as a minimization problem over finite-dimensional sets. The variational structure is inherited by the corresponding variational interpretation of the underlaying PDE problems, thereby facilitating the use of tools from PDE theory for the error analysis of the FEM. Reed and Hill propounded a new class of FEM in 1971 which was the discontinuous Galerkin finite element method for the numerical solution of the nuclear transport PDE problem. This involves a linear first-order hyperbolic partial differential equation (PDE). In the area of parabolic problems, Nitsche's seminal work on weak imposition of essential boundary conditions for FEM, allowed for finite element solution spaces that do not satisfy the essential boundary conditions. DG methods were first propounded and analyzed in the early 1970s as a technique to numerically solve partial differential equations. In 1973, Reed and Hill [1], introduced a DG method to solve the hyperbolic neutron transport equation. The origin of the DG method for parabolic problems cannot be traced back to a single publication as features such as jump penalization in the modern sense were developed gradually. A more complete account of the historical development and an introduction to DG methods for parabolic problems is given in a publication by P. Lax and N. Milgram [2]. A number of research directions and challenges on DG methods are collected in the proceedings volume edited by Cockburn, Karniadakis, and Shu [3]. The discontinuous Galerkin (DG) method has been extensively studied and applied to a wide range of parabolic problems. Beatrice Riviere [4], offered the Discontinuous Galerkin methods for solving elliptic and parabolic equations with theory and implementation. Jan S. Hesthaven and Tim Warburton [5], described the Nodal Discontinuous Galerkin Methods;

Algorithms, Analysis and also described various applications of the method. P. E. Lewis and J.P. Ward [6], provided a general introduction to the Finite Element Method. D.N. Arnoldis [7], presented an interior penalty finite element method with discontinuous elements. R. Becker, P. Hansbo, and, M.G. Larson [8], provided the energy norm in the case of a posteriori error estimation for discontinuous Galerkin methods. C. Carstensen, T. Gudi, and M. Jensen [9], included the error estimate with discontinuous Galerkin (DG) FEM to unifying the theory of a posteriori error approximation. B. Cockburn [10], published a book on Discontinuous Galerkin methods for convection-dominated problems and showed in the case of higher-order, the methods has been existed vast information. B. Cockburn, G.E. Karniadakis, and C.-W [11] explained this DG method in the perception of the theory, computation, and applications of the problem. E.H. Georgoulis [12], comprised the shape-regular meshes on discontinuous Galerkin (DG) FEM. Sjodin and Bjorn [13], demonstrated the conceptual difference among FEM, FDM, and, FVM to give a clear idea about the FEM, FDM, and FVM. B. Cockburn, G. E. Karniadakis and, C.-W. Shu (eds.) [14], represented the theoretical and computational framework of the DG method. I. Babuˇska [15], provided the mathematical justification of the DG method by applying the Lagrangian multipliers. S. Brenner and L. Scottfor [16], established the mathematical structure of Finite Element Methods. B. Cockburn, G. Kanschat, and, D. Schötzau [17], symbolized the local discontinuous Galerkin method for the Oseen equations. Hailiang Liu, Jue Yan [18, 19], introduced the Direct Discontinuous Galerkin (DDG) method for diffusion with interface corrections. I. Babuˇska [20], provided some information for applying Lagrangian multipliers on the finite element. The development of the DG method has been found in [21, 22]. The main aim of this article is to design a suitable error approximation of the second order parabolic differential equation by using the discontinuous Galerkin (DG) finite element method.

2. Model problem

Consider a polygonal domain in $\mathbb{R}^d$ by Ω, where $d = 2 \; or \; 3$. The sides of the boundary $\partial\Omega$ of the domain are divided into two disjoint sets Γ_D and Γ_N. Let n be the unit normal vector to the boundary exterior to Ω. For f given in $L^2(\Omega)$, g_D given in $H^1(\Gamma_D)$, and g_N given in $L^2(\Gamma_N)$. Now, consider the parabolic problem:

$$\nabla \cdot (\nabla u) + cu = f \; in \; \Omega, \quad \ldots\ldots\ldots (1)$$

$$u = g_D \; on \; \Gamma_D, \quad \ldots\ldots\ldots (2)$$

$$\nabla u \cdot n = g_N \; on \; \Gamma_N, \quad \ldots\ldots\ldots (3)$$

Where, c is a non-negative scalar function. The second equation (2) is a Dirichlet boundary condition. The value of the solution is defined on Γ_D. The third equation (3) is a Neumann boundary condition. The normal derivative or flux is defined on Γ_N.

From the equation (1) to have,

$$\nabla \cdot (\nabla u) + cu = f$$

$$\Rightarrow \nabla^2 u + cu = f$$

$$\Rightarrow \nabla^2 (u_D + w) + c(u_D + w) = f$$

$$\Rightarrow \nabla^2 u_D + \nabla^2 w + cu_D + c\,w = f$$

$$\Rightarrow \nabla^2 u_D v + \nabla^2 w\, v + cu_D v + c\,w\,v = fv$$

$$\Rightarrow \int_\Omega (\nabla^2 u_D v + \nabla^2 w\, v + cu_D v + c\,w\,v) = \int_\Omega fv$$

$$\Rightarrow \int_\Omega \nabla^2 u_D v + \int \nabla^2 w\, v + \int_\Omega cu_D v + \int_\Omega c\,w\,v = \int_\Omega fv$$

$$\Rightarrow v\nabla u_D - \int_\Omega \nabla v.\nabla u_D + v\nabla w - \int_\Omega \nabla v.\nabla w + \int_\Omega (cu_D v + c\,w\,v) = \int_\Omega fv$$

$$\Rightarrow -\int_\Omega \nabla v.\nabla u_D - \int_\Omega \nabla v.\nabla w + \int_\Omega cu_D v + \int_\Omega c\,w\,v = \int_\Omega fv$$

$$\Rightarrow -\int_\Omega (\nabla v.\nabla u_D - cu_D v) - \int_\Omega (\nabla v.\nabla w - c\,w\,v) = \int_\Omega fv$$

$$\Rightarrow -\int_{\Omega} (\nabla v . \nabla w - c\, w\, v) = \int_{\Omega} f v + \int_{\Omega} (\nabla v . \nabla u_D - c u_D v)$$

The solution u of the problems (1) to (3) is called the weak solution.

3. Jump and averages

Γ_h is the set of interior edges of the subdivision $\mathcal{E}_h$. A unit normal vector $n_e$with each edge e.The n_e is taken to be the unit outward vector normal to $\partial\Omega$ if e is on the boundary $\partial\Omega$. If v belongs to $H^1(\mathcal{E}_{h)})$, the trace of v along any side of one element E is well defined. If two elements E_1^e and E_2^e are neighbours and share one common side e, there are two traces of v along e. If add two elements E_1^e & E_2^e and then divide by 2, then obtain an average. On the other hand, if subtract them, get a jump for. Let the normal vector $n_e$is oriented from E_1^e to E_2^e. The average is,

$$\{v\} = \frac{1}{2}(v|_{E_1^e}) + \frac{1}{2}(v|_{E_1^e}), \quad \forall e = \partial E_1^e \cap \partial E_2^e.$$

The jump is,

$$[v] = (v|_{E_1^e}) - (v|_{E_1^e}), \quad \forall e = \partial E_1^e \cap \partial E_2^e.$$

In one-dimensional case, the jump and average can be written as

$$\{v\} = [v] = (v|_{E_1^e}) \quad \forall e = \partial E_1^e \cap \partial\Omega.$$

Because in this case, sides belong to the boundary $\partial\Omega$.

4. Variational formulation

Let two bilinear forms $J_0^{\delta_0,\gamma_0}, J_1^{\delta_1,\gamma_1} : H^s(\mathcal{E}_h) \times H^s(\mathcal{E}_h) \to \mathbb{R}$ that penalize the jump of the functional values and the jump of the normal derivatives values:

$$J_0^{\delta_0,\gamma_0}(v,w) = \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{\delta_e^0}{|e|^{\gamma_0}} \int_e [v][w]$$

$$J_1^{\delta_1,\gamma_1}(v,w) = \sum_{e\in\Gamma_h} \frac{\delta_e^1}{|e|^{\gamma_1}} \int_e [\nabla v . n_e][\nabla w . n_e]$$

Where the parameters δ_e^0 and δ_e^1 are penalty parameters and non-negative real numbers. The indexes γ_0 and γ_1 are positive numbers. They depend on the dimension d. It is known that the notation $|e|$ simply means the length of e in $2D$ and the area of e in $3D$. Then, clearly have,

$$\forall e \subset \partial E, |e| \le h_E^{d-1} \le h^{d-1}.$$

Now, define the DG bilinear forms $b_\varepsilon : H^s(\mathcal{E}_h) \times H^s(\mathcal{E}_h) \to \mathbb{R}$
Which can be written as

$$b_\varepsilon(v,w) = \sum_{E\in\mathcal{E}_h} \int_E \nabla v \cdot \nabla w + \int_\Omega cvw - \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\nabla v \cdot n_e\}[w] + \varepsilon \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\nabla w \cdot n_e\}[v] + J_0^{\delta_0,\gamma_0}(v,w) + J_1^{\delta_1,\gamma_1}(v,w)$$

The bilinear form b_ε holds another parameter ϵ whose values may be taken as $-1, 0$ *or* 1.These follows the symmetric property.

1) b_ε is symmetric if $\epsilon = -1$
2) b_ε is non-symmetric if $\epsilon = 1$ *or* 0

The linear form is denoted by $L(v)$ and defined by:

$$L(v) = \int_\Omega fv + \varepsilon \sum_{e\in\Gamma_D} \int_e (\nabla v \cdot n_e + \frac{\delta_e^0}{|e|^{\gamma_0}} v)\, g_D + \sum_{e\in\Gamma_N} \int_e v g_N$$

If the functions belong to $H^s(\mathcal{E}_h)$ for any $s > \frac{3}{2}$, Cauchy-Schwarz's inequality and trace inequalities imply that all integral terms in the forms defined above make sense.

Then the general DG variational formulation of the problem (1) to the problem (3) is as follows:

Get $\boldsymbol{u}$ in $\boldsymbol{H^s}(\mathcal{E}_h), s > \frac{3}{2}$, such that

$$\forall v \in H^s(\mathcal{E}_h), b_\varepsilon(u, v) = L(v) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

5. Consistency

Assume that the weak solution $\boldsymbol{u}$ of the problem (1) to the problem (3) belongs to $\boldsymbol{H^s}(\mathcal{E}_h)$; then $\boldsymbol{u}$ satisfies the variational problem (4). Conversely, if $u \in H^1(\Omega) \cap H^s(\mathcal{E}_h)$ satisfies (4), then u is the solution of problem (1) *to* (3). Now, prove that if the solution $\boldsymbol{u}$ of (1)-(3) belongs to $\boldsymbol{H^s}(\Omega)$, then it also solves (4). To prove this, let $\boldsymbol{v}$ be an element in $\boldsymbol{H^s}(\mathcal{E}_h)$.

From equation (1) to have,

$$\nabla \cdot (\nabla u) + cu = f$$

Multiply by a weight function $\boldsymbol{v}$

$$\Rightarrow \nabla \cdot (\nabla u)v + cuv = fv$$

By integrating both side on the elements $\boldsymbol{E}$ to get

$$\Rightarrow \int_E \nabla \cdot (\nabla u)v + \int_E cuv = \int_E fv$$

By using Green's theorem in first term of right-hand side.

$$-\int_E \nabla u \cdot \nabla v + \int_{\partial E} \nabla u \cdot n_E v + \int_E cuv = \int_E fv$$

$$-\int_E (\nabla u \cdot \nabla v - cuv) + \int_{\partial E} \nabla u \cdot n_E v = \int_E fv$$

$\boldsymbol{n_E}$ is the outward normal to $\boldsymbol{E}$. Sum over all elements and then switch to the normal vectors $\boldsymbol{n_E}$. Then,

$$\sum_{E \in \mathcal{E}_h} \int_{\partial E} \nabla u \cdot n_E v = \sum_{e \in \Gamma_h} \int_e [\nabla u \cdot n_e v] + \sum_{e \in \partial\Omega} \int_e \nabla u \cdot n_e v$$

By the rules of the solution $\boldsymbol{u}$, to have,

$$\nabla u \cdot n_E = \{\nabla u \cdot n_E\}$$

obtain the following equation

$$-\sum_{E \in \mathcal{E}_h} \int_E (\nabla u \cdot \nabla v - cuv) + \sum_{e \in \Gamma_h} \int_e [\nabla u \cdot n_e v] + \sum_{e \in \partial\Omega} \int_e \nabla u \cdot n_e v = \int_\Omega fv$$

By applying the Neumann boundary condition (3) on it,

$$-\sum_{E \in \mathcal{E}_h} \int_E (\nabla u \cdot \nabla v - cuv) + \sum_{e \in \Gamma_h} \int_e \{\nabla u \cdot n_e\}[v] + \sum_{e \in \partial\Omega} \int_e \nabla u \cdot n_e v = \int_\Omega fv + \sum_{e \in \Gamma_N} \int_e g_N v$$

Now, adding $\varepsilon \sum_{e \in \Gamma_D} \int_e (\nabla v \cdot n_e) u$ and $\sum_{e \in \Gamma_D} \frac{\delta_e^0}{|e|^{\gamma_0}} \int_e uv$ to both sides and apply the Dirichlet boundary condition (2),

$$-\sum_{E \in \mathcal{E}_h} \int_E (\nabla u \cdot \nabla v - cuv) + \sum_{e \in \Gamma_h} \int_e \{\nabla u \cdot n_e\}[v] + \sum_{e \in \partial\Omega} \int_e \nabla u \cdot n_e v + \varepsilon \sum_{e \in \Gamma_D} \int_e (\nabla v \cdot n_e) u + \sum_{e \in \Gamma_D} \frac{\delta_e^0}{|e|^{\gamma_0}} \int_e uv$$
$$= \int_\Omega fv + \sum_{e \in \Gamma_N} \int_e g_N v + \varepsilon \sum_{e \in \Gamma_D} \int_e (\nabla v \cdot n_e) g_D + \sum_{e \in \Gamma_D} \frac{\delta_e^0}{|e|^{\gamma_0}} \int_e g_D v$$

Since the jumps $[u] = [\nabla u \cdot n_e]$ are zero, then this are same to the equation (4).

Conversely, by taking $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\boldsymbol{E})$. The equation (4) can be written as

$$\sum_{E\in\mathcal{E}_h}\int_E \nabla u \cdot \nabla v + \int_\Omega cuv = \int_\Omega fv \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

which instantly yields in the distributional sense, $\forall \boldsymbol{E} \in \mathcal{E}_h$,

$$\nabla \cdot \nabla u + cu = f \quad in\ E$$

Again let $\boldsymbol{e}$ be an interior edge. Also let E_e^1 and E_e^2 be the two elements which is adjacent to $\boldsymbol{e}$.

By taking $\boldsymbol{v} \in H_0^2(E_e^1 \cup E_e^2)$, extend it by zero over the residual domain. Now, if multiply (5) by $\boldsymbol{v}$ and then use Green's theorem,

$$-\int_{E_e^1\cup E_e^2} \nabla u \cdot \nabla v + \int_e [\nabla u \cdot n_e] v + \int_{E_e^1\cup E_e^2} cuv = \int_{E_e^1\cup E_e^2} fv$$

On the contrary, if $[\boldsymbol{v}] = \boldsymbol{0}$, then from (4) we have,

$$-\int_{E_e^1\cup E_e^2} \nabla u \cdot \nabla v + \int_{E_e^1\cup E_e^2} cuv = \int_{E_e^1\cup E_e^2} fv$$

Therefore, we have,

$$\forall v \in H_0^2(E_e^1 \cup E_e^2), \qquad \int_e [\nabla u \cdot n_e] v = 0$$

Which we can be written as $[\nabla u \cdot n_e]|_e = 0$ in $L^2(e)$. As this contains for all $\boldsymbol{e}$, it implies that $\nabla \cdot \nabla u \in L^2(\Omega)$, and therefore we usually obtain,

$$\nabla \cdot \nabla u + cu = f \quad in\ \Omega \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$$

We multiply (6) by a function $\boldsymbol{v}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ to regain the Dirchelet boundary conditions. Then we apply Green's theorem and compare with (4):

$$\sum_{e\in\Gamma_D}\int_e (\nabla v \cdot n_e)(u - g_D) = 0$$

Which is true for all $\boldsymbol{v} \in H^2(\Omega) \cap H_0^1(\Omega)$, we get $\boldsymbol{u} = \boldsymbol{g_D}$ on $\boldsymbol{\Gamma_D}$. Once for all, by taking $\boldsymbol{v} \in H^2(\Omega)$, $v|_{\Gamma_D} = 0$, we have

$$\sum_{e\in\Gamma_N}\int_e (\nabla u \cdot n_e) v = \sum_{e\in\Gamma_N}\int_e g_N v$$

Which gives a Nuemann boundary Condition and equal to (3).

6. Coercivity of bilinear forms

From the definition of coercivity we know that if there is a positive constant $\boldsymbol{\tau}$ such that $\forall v \in V, \tau \|v\|_V^2 \le b_\varepsilon(v, v)$, then a bilinear form is defined on a normal linear space $\boldsymbol{V}$ with norm $\|\cdot\|_V$ is coercive.

We know the DG bilinear form

$$b_\varepsilon(v, v) = \sum_{E\in\mathcal{E}_h}\int_E (\nabla v)^2 + \int_\Omega cv^2 + (\varepsilon - 1)\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\nabla v \cdot n_e\}[v] + J_0^{\delta_0,\gamma_0}(v, v)$$

Now we define the energy norm on $\mathcal{D}_k(\mathcal{E}_h)$:

$$\|v\|_\varepsilon = \Big(\sum_{E\in\mathcal{E}_h}\int_E \nabla v . \nabla v + \int_\Omega cv^2 + J_0^{\delta_0,\gamma_0}(v, v)\Big)^{\frac{1}{2}}$$

If $\delta_0^e > 0$ for all $\boldsymbol{e}$, then it is easy to check that it is actually a norm, immediately have the coercivity property which is satisfied for $\epsilon = 1$. Where $\tau = 1$ is the coercivity constant. Actually,

$$\forall v \in \mathcal{D}_k(\mathcal{E}_h), \qquad \|v\|_\varepsilon^2 = b_\varepsilon(v, v).$$

In case of $\epsilon = -1$ or $\epsilon = 0$, we have using Cauchy-Schwartz's inequality an upper bound of the term $\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\nabla v \cdot n_e\}[v]$:

$$\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\nabla v \cdot n_e\}[v] \le \sum_{e\in\Gamma_h\cup\Gamma_D}\|\{\nabla v \cdot n_e\}\|_{L^2(e)}\|[v]\|_{L^2(e)} \le \sum_{e\in\Gamma_h\cup\Gamma_D}\|\{\nabla v \cdot n_e\}\|_{L^2(e)}\left(\frac{1}{|e|^{\gamma_0}}\right)^{\frac{1}{2}-\frac{1}{2}}\|[v]\|_{L^2(e)}$$

Consider the average of the normal derivatives for an interior edge e which is shared by the elements E_1^e and E_2^e, then to get,

$$\|\{\nabla v \cdot n_e\}\|_{L^2(e)} \le \frac{1}{2}\left\|(\nabla v \cdot n_e)|_{E_1^e}\right\|_{L^2(e)} + \frac{1}{2}\left\|(\nabla v \cdot n_e)|_{E_2^e}\right\|_{L^2(e)}$$

By using trace inequality,

$$\|\{\nabla v \cdot n_e\}\|_{L^2(e)} \le \frac{1}{2}\left\|(\nabla v \cdot n_e)|_{E_1^e}\right\|_{L^2(e)} + \frac{1}{2}\left\|(\nabla v \cdot n_e)|_{E_2^e}\right\|_{L^2(e)} \le \frac{C_t}{2}h_{E_1^e}^{-\frac{1}{2}}\|\nabla v\|_{L^2(E_1^e)} + \frac{C_t}{2}h_{E_2^e}^{-\frac{1}{2}}\|\nabla v\|_{L^2(E_2^e)}$$

Using $\forall e \subset \partial E, |e| \le h_E^{d-1} \le h^{d-1}$, then,

$$\begin{aligned}\int_e \{\nabla v \cdot n_e\}[v] &\le \frac{C_t}{2}|e|^{\frac{\gamma_0}{2}}h_{E_1^e}^{-\frac{1}{2}}\|\nabla v\|_{L^2(E_1^e)} + h_{E_2^e}^{-\frac{1}{2}}\|\nabla v\|_{L^2(E_2^e)}\left(\frac{1}{|e|^{\gamma_0}}\right)^{\frac{1}{2}}\|[v]\|_{L^2(e)}\\ &\le \frac{C_t}{2}\left(h_{E_1^e}^{\frac{\gamma_0}{2}(d-1)-\frac{1}{2}} + h_{E_2^e}^{\frac{\gamma_0}{2}(d-1)-\frac{1}{2}}\right)\left(\|\nabla v\|_{L^2(E_1^e)}^2 + \|\nabla v\|_{L^2(E_2^e)}^2\right)^{\frac{1}{2}}\left(\frac{1}{|e|^{\gamma_0}}\right)^{\frac{1}{2}}\|[v]\|_{L^2(e)}\\ &\le C_t\left(\|\nabla v\|_{L^2(E_1^e)}^2 + \|\nabla v\|_{L^2(E_2^e)}^2\right)^{\frac{1}{2}}\left(\frac{1}{|e|^{\gamma_0}}\right)^{\frac{1}{2}}\|[v]\|_{L^2(e)}.\end{aligned}$$

Assume with the help of 'without loss of generality' that is $h \le 1$ and γ_0 satisfies the condition $\gamma_0(d-1) \ge 1$, then a similar bound is obtained where e is a boundary edge. Let n_0 denote the maximum number of neighbours of an element. For example, for a confirming mesh, $n_0 = 3$ and $n_0 = 4$ for a triangle and a quadrilateral.

$$\begin{aligned}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\nabla v \cdot n_e\}[v] &\le C_t\left(\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{1}{|e|^{\gamma_0}}\|[\nabla v]\|_{L^2(e)}^2\right)^{\frac{1}{2}}\Big(\sum_{e\in\Gamma_h}\|\nabla v\|_{L^2(E_1^e)}^2 + \|\nabla v\|_{L^2(E_2^e)}^2 + \sum_{e\in\Gamma_D}\|\nabla v\|_{0,E_1^e}^2\Big)\\ &\le C_t\sqrt{n_0}\Big(\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{1}{|e|^{\gamma_0}}\|[\nabla v]\|_{L^2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{E\in\varepsilon_h}\|\nabla v\|_{L^2(E)}^2\Big)^{\frac{1}{2}}\end{aligned}$$

For $\mu > 0$, we use Young's inequality. Then,

$$\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\nabla v \cdot n_e\}[v] \le \frac{\mu}{2}\sum_{E\in\varepsilon_h}\|\nabla v\|_{L^2(E)}^2 + \frac{C_t^2 n_0}{2\mu}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{1}{|e|^{\gamma_0}}\|[\nabla v]\|_{L^2(e)}^2$$

In this way, gain a lower bound for $b_\varepsilon(v, v)$:

$$b_\varepsilon(v, v) \ge \left(1 - \frac{\mu}{2}|1-\varepsilon|\right)\sum_{E\in\varepsilon_h}\|\nabla v\|_{L^2(E)}^2 + \sum_{e\in\Gamma_h\cup\Gamma_D}\frac{\delta_e^0 - \frac{C_t^2 n_0}{2\mu}|1-\varepsilon|}{|e|^{\gamma_0}}\|[v]\|_{L^2(e)}$$

On this occasion, choose for instance, $\mu = 1$ *and* $\mu = \frac{1}{2}$ if $\varepsilon = 0$ *and* $\varepsilon = -1$ and also choose δ_e^0 large enough (for example, $\delta_e^0 \ge (C_t^2 n_0)$ for $\epsilon = 0$ and $\delta_e^0 \ge (2C_t^2 n_0)$ for $\varepsilon = -1$), then we have the coercivity result such that

$$b_\varepsilon(v, v) \ge \|v\|_\varepsilon^2$$

Now, summarize the results above. So,

1) b_{+1} is coercive;
2) b_{-1} and b_0 are coercive if $\gamma_0(d-1) \ge 1$ and if δ_e^0 is bounded below by a constant δ_e^*.

7. Existence and uniqueness of DG solution

Existence is equivalent to uniqueness because $\forall\, \boldsymbol{v} \in \mathcal{D}_k(\mathcal{E}_h),\ \boldsymbol{b}_\varepsilon(\boldsymbol{U}_h, \boldsymbol{v}) = \boldsymbol{L}(\boldsymbol{v})$ is a linear problem and also has finite dimension. Suppose that $\boldsymbol{u}_h^1$ and $\boldsymbol{u}_h^2$ are two solutions. The sub-triplication $\boldsymbol{w}_h = \boldsymbol{u}_h^1 - \boldsymbol{u}_h^2$ verifies that $\boldsymbol{b}_\varepsilon(\boldsymbol{w}_h, \boldsymbol{w}_h) = \boldsymbol{0}$.

By using coercivity result $\boldsymbol{\tau}\, \|\boldsymbol{v}\|_V^2 \le \boldsymbol{b}_\varepsilon(\boldsymbol{v}, \boldsymbol{v})$, to obtain,

$$\|\boldsymbol{w}_h\|_\varepsilon = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{w}_h = \boldsymbol{0} \text{ since } \|\cdot\|_\varepsilon \text{ is a norm}$$

Actually, $\boldsymbol{w}_h$ is piecewise constant on each element $\boldsymbol{E} \in \mathcal{E}_h$. To prove that $\boldsymbol{w}_h$ is universally constant in Ω, to set up a test function $\boldsymbol{v}$ on a given element $\boldsymbol{E}$ such that the expression $\int_e \nabla\boldsymbol{v} \cdot \boldsymbol{n}_e$ is given on one edge of $\boldsymbol{E}$ and vanishes on the other edges.

8. Error analysis

Suppose that, the exact solution $\boldsymbol{u}$ belongs to $\boldsymbol{H}^s(\mathcal{E}_h)$ and try to prove that DG solution converges to the exact solution. To estimate error, use the $\boldsymbol{L}^2$ norm that is proving an error estimate in the $\boldsymbol{L}^2$ norm. To prove this, apply the Aubin-Nitsche lift method which is used in the analysis of the classical finite element method to the DG method. This method works well if the process is symmetric. For simplicity, suppose that the entire boundary is a Dirichlet boundary, that is, $\partial\Omega = \boldsymbol{\Gamma}_D$. Also suppose that the domain is convex and the solution is to be the dual problem

$$\nabla \cdot (\nabla\zeta) + c\zeta = u - U_h \quad in\ \Omega,$$

$$\zeta = 0 \quad on\ \partial\Omega$$

Which is belongs to $\boldsymbol{H}^2(\Omega)$ with continuous dependence on $-\boldsymbol{U}_h$:

$$\|\zeta\|_{H^2(\Omega)} \le C\|u - U_h\|_{L^2(\Omega)}$$

So,

$$\|U_h - u\|_{L^2(\Omega)}^2 = \int_e (U_h - u)^2 = \int_\Omega (\nabla \cdot (\nabla\zeta) + c\zeta)\,(U_h - u)$$

Let $\boldsymbol{\omega} = \boldsymbol{U}_h - \boldsymbol{u}$ and then integrate by parts on each element
Therefore,

$$\|\omega\|_{L^2(\Omega)}^2 = \sum_{E\in\varepsilon_h} \int_E (\nabla\zeta.\nabla\omega + c\zeta\omega) - \sum_{E\in\varepsilon_h} \int_{\partial E} (\nabla\zeta.n_e)\omega$$

As $\boldsymbol{\zeta} \in \boldsymbol{H}^2(\Omega)$, to obtain,

$$\|\omega\|_{L^2(\Omega)}^2 = \sum_{E\in\varepsilon_h} \int_E (\nabla\zeta.\nabla\omega + c\zeta\omega) - \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\nabla\zeta.n_e\}[\omega]$$

The difference of orthogonal equation is

$$\forall v \in \mathcal{D}_k(\mathcal{E}_h), \qquad b_\varepsilon(U_h - u, v) = 0$$

From the equation above:

$$\forall v \in \mathcal{D}_k(\mathcal{E}_h)$$

$$\|\omega\|_{L^2(\Omega)}^2 = \sum_{E\in\varepsilon_h} \int_E (\nabla(\zeta - v).\nabla\omega + c(\zeta - v)\omega) - \varepsilon \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\nabla\zeta.n_e\}[\omega] - \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\nabla\zeta.n_e\}[\omega]$$
$$+ \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\nabla\zeta.n_e\}[v] - J_0^{\delta_0,\gamma_0}(\omega, v) = B_1 + \cdots + B_5$$

Suppose $\boldsymbol{v} = \tilde{\boldsymbol{\zeta}}$, a continuous interpolant of $\boldsymbol{\zeta}$ of degree $\boldsymbol{k}$. Yield that, such an interpolant lies. In that case, observe that $\tilde{\boldsymbol{\zeta}} = \boldsymbol{0}$ on the boundary $\partial\Omega$.

9. Conclusion

This paper has investigated the error of the numerical solution by applying the Discontinuous Galerkin finite element method for the second order parabolic differential equation. We considered discontinuous Galerkin finite element approximations of a model scalar linear parabolic equation. It is a different and straightforward approach to seek error analysis from all other finite element schemes which are given in the literature. The technique used in this paper can also be extended to obtain the $L^2(\Omega)$ error estimate of the time dependent and higher order problems with the optimal order of convergence.

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