

Analysis of Associated Stiffening Effect of Shear Deformation on Geometric Stiffness Matrix

Mingxuan Guo, Jun Dong*, Jingshu Lin, Yao Zhang

Beijing Higher Institution Engineering Research Center of Structural Engineering and New Materials, Beijing University of Civil Engineering and Architecture, Beijing, China.

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***Corresponding author:** Jun Dong, Beijing Higher Institution Engineering Research Center of Structural Engineering and New Materials, Beijing University of Civil Engineering and Architecture, Beijing, China.

Abstract

The influence of shear deformation on the geometrical stiffness matrix in the stability problem of compression bar is investigated in this paper. On the basis of deriving the element stiffness matrix and geometric stiffness matrix by energy method, the shear deformation is introduced, and the influence of bending, shear, flexural shear, and compression flexural shear actions on the correction of geometric stiffness matrix and the final critical load are discussed in detail. Furtherly, the concepts of associated stiffening and associated softening for stability problems are proposed by analogy with other scholars' ideas. The results show that the bending term and shear term soften the compression bar, and the associated terms such as the bending-shear accompanying term and compression-bending-shear accompanying term harden first and then soften. It is very important for practical engineering to make reasonable use of the associated stiffening effect of shear deformation.

Keywords

Shear Deformation, Compression Bar, Stability, Geometric Stiffness Matrix, Associated Stiffening Effect

1. Introduction

Many cases in engineering practice can be abstracted into a compression bar model to study, such as the upper chord and some web members of truss bridge [1], the buried pile foundation [2], the extended crane [3], the beam and the tower of a cable-stayed bridge [1], etc. The most important problem of this kind of compression bar model is its stability, that is, the calculation of critical load. Regarding the calculation of critical load, the most classical finite element method is to introduce a geometric stiffness matrix [1, 4]. This method can be seen as a way to visually equivalent the reduction effect of the slender rod on the critical load of the section to the reduction of each element in the original stiffness matrix by taking into account the axial pressure work along the longitudinal direction in the process of deriving the element stiffness matrix from the energy principle. This method is simple and practical, but the influence of shear deformation is ignored in the equation derivation process, so the calculation results of this pressure rod element may be inaccurate under the new combined cross-section [5, 6].

Based on the assumption of bending angle and shear angle, a compression rod element considering shear deformation is constructed in this paper. The changes in the geometric stiffness matrix of the improved compression rod element compared with the previous ones are discussed. The concepts of stiffening effect and softening effect on geometric stiffness matrix are put forward, and the influence of flexural and compressive flexural terms on critical loads is discussed. The discussion has certain significance for the stability problem, especially for the stability problem where the assumption of thin beams such as composite section beams and composite section beams is no longer applicable.

2. Basic Principles and Assumptions

2.1 The assumption of bending angle and shear angle

For the construction of finite elements, fundamental assumptions are very important. Here, with reference to the construction method of the Timoshenko beam element, it is assumed that the Angle of rotation of the element after bending is composed of two parts, namely, the bending Angle and the shear Angle. Among them, the bending Angle represents the angle θ caused by the bending moment action, and the shear Angle represents the Angle γ caused by the shearing force F_s action, then if the total deflection line equation is y , then [7]:

$$y' = \frac{dy}{dx} = \theta + \gamma \tag{1}$$

In addition, it can be seen from the mechanics of materials [8] that the θ is the slope of the deflection equation y_m caused by the bending moment M (m represents the deflection caused by bending), while the γ is related to the shear force F_s . Note the symbolic relationship between the shear force at the left end of the member and γ as shown in Figure 1 (the opposite sign at the right end), then $y' = y'_m - \frac{kF_s}{GA}$.

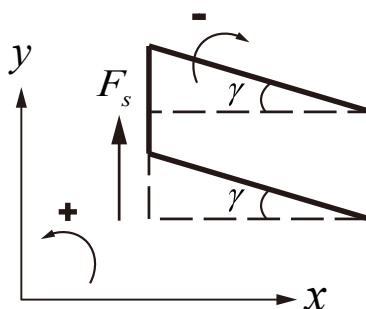


Figure 1. The relationship between F_s and γ .

Further, we set $\frac{kEI}{GA l^2}$ as μ [5] and the deflection equation of the beam is combined with equation (1), then:

$$y' = y'_m - \frac{k dM}{GA dx} = y'_m + \mu l^2 y''_m \tag{2}$$

However, if considering cross-section dislocation and equation (1), then:

$$y' = \theta + \gamma = \theta - \frac{k F_s}{GA} = \theta - \frac{k dM}{GA dx} + \frac{k N dy}{GA dx} \tag{3}$$

Sorted into:

$$y' = \frac{(\theta - \frac{k dM}{GA dx})}{(1 - \frac{k N}{GA})} = \frac{(y'_m + \mu l^2 y''_m)}{(1 - \frac{k N}{GA})} \approx \frac{(y'_m + \mu l^2 y''_m)}{(1 - \frac{k N_e}{GA})} = (1 + \frac{\frac{k N_e}{GA}}{1 - \frac{k N_e}{GA}})(y'_m + \mu l^2 y''_m) \tag{4}$$

Here, following the classical literature method from Timoshenko, N is approximately replaced by N_e , where N_e is the value of Euler's critical force under the corresponding boundary conditions.

2.2 The element stiffness matrix derived by energy method

First, the element is constructed according to the basic assumptions of 2.1 as shown in Figure 2, where it is specified that the rotation Angle is positive in a counterclockwise direction (x turns to y). Then, after considering the work done by the longitudinal force, the structural potential energy E_p can be listed as:

$$E_p = V_\epsilon + V_p + V_Q = \frac{1}{2} \int_0^l EI(y'')^2 dx - \frac{F}{2} \int_0^l (y')^2 dx - Fy \tag{5}$$

E_p is the total potential energy of the system, V_ϵ is the elastic strain energy of the system, V_p is the potential energy of the longitudinal force, V_Q is the potential energy of the rod end force, and y is the deflection equation of the pressure rod element. When the stable equilibrium condition is reached, according to the potential energy standing value principle δE_p is 0 and introduced y is $\sum_{i=1}^n \delta_i^e \phi_i$, substituted to finish:

$$\bar{F}_i^e = \sum_{j=1}^4 \bar{k}_{ij} \bar{\delta}_j^e - \sum_{j=1}^4 \bar{s}_{ij} \bar{\delta}_j^e \quad (i=1,2,3,4, j=1,2,3,4) \tag{6}$$

$$\bar{k}_{ij} = \int_0^l EI \phi_i'' \phi_j'' dx \quad \bar{s}_{ij} = F \int_0^l \phi_i' \phi_j' dx \tag{7}$$

If the axial deformation is not considered, the column vector of rod end displacement and rod end force is expressed as:

$$\bar{\delta}^e = (\bar{v}_i^e \quad \bar{\varphi}_i^e \quad \bar{v}_j^e \quad \bar{\varphi}_j^e)^T, \bar{F}^e = (\bar{F}_{Si}^e \quad \bar{M}_i^e \quad \bar{F}_{Sj}^e \quad \bar{M}_j^e)^T \tag{8}$$

Where, \bar{k}_{ij} and \bar{s}_{ij} are the expressions of each element in the original stiffness matrix and the geometric stiffness matrix. The matrix K and s are the original stiffness matrix and the geometric stiffness matrix.

Finally, the stability equation can be solved with the direct stiffness method as follows:

$$|K - s| = 0 \tag{9}$$

After a series of F are obtained from equation (9), the minimum positive solution is the critical load F_{cr} .

3. Theoretical derivation

3.1 Geometric stiffness matrix without considering the cross-section dislocation

Let's first deduce the case without considering the cross-section dislocation. According to 2.2, equation (2) is substituted into equation (7), then compared with the traditional compression rod element (without considering γ), the derivation part only \bar{s}_{ij} changes. Let each element of the changed geometric stiffness matrix be \bar{s}_{ij}' :

$$\begin{aligned} \bar{s}_{ij}' &= F \int_0^l y_i' y_j' dx = F \int_0^l (y_{mi}' + \mu l^2 y_{mi}''') (y_{mj}' + \mu l^2 y_{mj}''') dx \\ &= F \int_0^l y_{mi}' y_{mj}' dx + F \int_0^l (y_{mi}' c_j + y_{mj}' c_i) dx + F \int_0^l c_i c_j dx \\ &= \bar{s}_{ij}^m + \bar{s}_{ij}^{ms} + \bar{s}_{ij}^s \end{aligned}$$

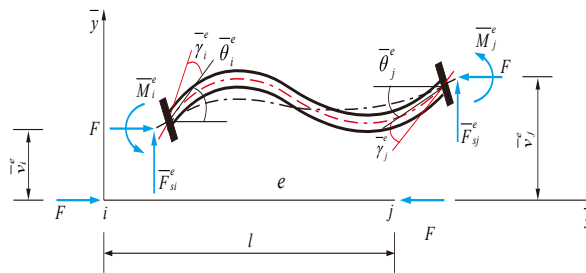


Figure 2. Improved Compression bar element.

Where, $c = \mu l^2 y_m'''$, $\bar{s}_{ij}^m = F \int_0^l y_{mi}' y_{mj}' dx$, $\bar{s}_{ij}^s = F \int_0^l c_i c_j dx$ and $\bar{s}_{ij}^{ms} = F \int_0^l (y_{mi}' c_j + y_{mj}' c_i) dx$. \bar{s}_{ij}^m represents the influence of bending on the geometric stiffness matrix, and the matrix formed by the combination is expressed as s^m ; \bar{s}_{ij}^s represents the influence of shear on the geometric stiffness matrix, and the matrix formed by the combination is expressed as s^s ;

\bar{s}_{ij}^{ms} represents the influence of flexural shear interaction on the geometric stiffness matrix. The matrix formed by their combination is expressed as s^{ms} .

According to the Timoshenko energy method, we assume y_m is $\sum_{i=1}^4 \varphi_i(x) = A + Bx + Cx^2 + Dx^3$, y_m represents the deflection equation produced by bending. The boundary conditions can be shown as follows:

$$\begin{cases} x = 0 : y = u_1, y' = \theta_1 \\ x = l : y = u_2, y' = \theta_2 \end{cases} \tag{10}$$

Solve it and expand into a 6×6 matrix (with axial deformation introduced) as follows:

$$s^m = F \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5l} & \frac{1}{10} & 0 & -\frac{6}{5l} & \frac{1}{10} \\ 0 & \frac{1}{10} & \frac{2l}{15} & 0 & -\frac{1}{10} & -\frac{l}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5l} & -\frac{1}{10} & 0 & \frac{6}{5l} & -\frac{1}{10} \\ 0 & \frac{1}{10} & -\frac{l}{30} & 0 & -\frac{1}{10} & \frac{2l}{15} \end{bmatrix} \tag{11}$$

$$s^s = F \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{144\mu^2}{l} & 72\mu^2 & 0 & -\frac{144\mu^2}{l} & 72\mu^2 \\ 0 & 72\mu^2 & 36\mu^2 l & 0 & 72\mu^2 & 36\mu^2 l \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{144\mu^2}{l} & -72\mu^2 & 0 & \frac{144\mu^2}{l} & -72\mu^2 \\ 0 & 72\mu^2 & 36\mu^2 l & 0 & -72\mu^2 & 36\mu^2 l \end{bmatrix} \tag{12}$$

$$s^{ms} = F \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{24\mu}{l} & -6\mu & 0 & \frac{24\mu}{l} & -6\mu \\ 0 & -6\mu & 0 & 0 & -6\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{24\mu}{l} & 6\mu & 0 & -\frac{24\mu}{l} & 6\mu \\ 0 & -6\mu & 0 & 0 & 6\mu & 0 \end{bmatrix} \tag{13}$$

Observing equations (11)-(13), it can be seen that equation (11) are consistent with the traditional conclusion [4]. It can be referred to as the pure bending term here. Because all elements in equation (12) are related to μ , which is a parameter introduced due to the shear deformation, we called it pure shear term. Equation (13) is a term that arising from shear force, so we can call it the bending-shear accompanying term (Similar to the concept of free vibration accompanying forced vibration problems in structural dynamics [13]), representing the influence term that accompanies the combined action of bending and shear.

3.2 Geometric stiffness matrix under the cross-section dislocation

If equation (3) is used to assume that, compared to 3.1, the improved \bar{s}_{ij}^n can be expressed as:

$$\bar{s}_{ij}'' = \bar{s}_{ij}^m + \bar{s}_{ij}^{ms} + \bar{s}_{ij}^s + \bar{s}_{ij}^{msn} = \left(\frac{1}{1 - \frac{kN_e}{GA}} \right)^2 \bar{s}_{ij}' \quad (14)$$

Among the equation, $\bar{s}_{ij}' = \bar{s}_{ij}^m + \bar{s}_{ij}^{ms} + \bar{s}_{ij}^s$, just like 3.1. $\bar{s}_{ij}^{msn} = \left(\frac{GA}{kN_e} \right)^2 \bar{s}_{ij}'$, \bar{s}_{ij}^{msn} is the accompanying term of geometric stiffness matrix concerned about compression bending shear effect.

4. Discussion and Analysis

4.1 The example of rigid frame

The example is taken from the structural mechanics [5] to calculate the critical load of the rigid frame as shown in the figure. The rigid frame structure is composed of three rods, each of which is l in length. The section of the rigid frame is rectangular, the width is b , the height is h , the depth-span ratio is $1/10$, E/G is $8/3$.

4.2 The effect of bending

Here, follow the method in 2.2 and set $\frac{Fl^2}{30EI}$ as β . the final stability equation as follows:

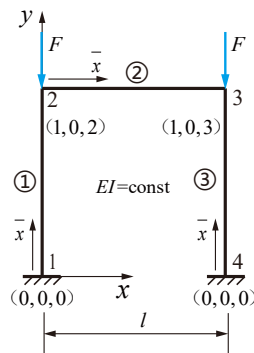


Figure 3. Example of rigid frame.

$$\begin{vmatrix} 24 - 72\beta & (6 - 3\beta)l & (6 - 3\beta)l \\ (6 - 3\beta)l & (8 - 4\beta)l^2 & 2l^2 \\ (6 - 3\beta)l & 2l^2 & (8 - 4\beta)l^2 \end{vmatrix} = 0 \quad (15)$$

Finally, F_{cr}^m equals $\frac{30\beta_{min}EI}{l^2}$. After calculation, β_{min} equals 0.248154219809881, so F_{cr} is $\frac{7.4446EI}{l^2}$, which is

0.89% larger than the exact solution.

The larger result here is mainly due to the difference between the deflection equation assumed by the energy method and the actual situation, which is equivalent to placing certain constraints on the actual deformation. Therefore, it indirectly increases the overall stiffness of the element, thus increasing the critical load.

By observing the whole formula form, it can be found that the essence of the geometric stiffness matrix is to consider a weakening on the basis of the ordinary beam element stiffness matrix. Under the pressure, the structure will produce large geometric deformation due to the p - δ effect, resulting in instability under the action of critical load. Since the critical load at this time is less than the compressive strength value at the time of failure, the engineering generally considers it as a weakening of beam structure under the axial pressure. By analogy with the stress stiffening effect and rotary softening effect [10, 11] proposed by previous scholars for similar problems, the loss of lateral stiffness (instability direction) of the compression rod structure under pressure can be called the stress softening effect. And because the cause of instability is the bending, it can be called the bending softening effect. s^m matrix is called a flexural softening matrix, and the generation

of this matrix is also the reason for the buckling of pressure rod [12].

4.3 The discussion of parameter μ

The μ of other depth-span ratios and other cross sections under different depth-span ratios can be calculated by referring to Lu Caifeng's paper. Some values of μ of common cross sections are given in the following table after calculation, and the data results are summarized in the following figure.

After considering the shear deformation, there will be coefficients μ and μ^2 in the matrix. According to the author's previous research, this coefficient is related to the cross-section type and the depth-span ratio [7]. The calculation process of μ for a rectangular section with a depth-span ratio of 1/10 is given here: $\mu = \frac{kEI}{GA l^2}$, and $I = \frac{bh^3}{12}$, $\frac{E}{G} = 2(1+\nu) = \frac{8}{3}$ (for a rectangular section beam), and the coefficient of shear uniformity coefficient k is 1.2, so when the depth-span ratio is $\frac{1}{10}$, the solution is: μ is 1/375.

The μ of other high-span ratios and other cross sections under different high-span ratios can be calculated by referring to C. Lu's paper [13]. Some values of μ of common cross sections are given in the following table after calculation, and the data results are summarized in the following figure.

Table 1. The value of type of common section and depth-span ratio

| Depth-span ratio | Section Type | | | | |
|------------------|--------------|----------|----------|----------|--------------|
| | Rectangular | Circle | Pipe* | I40a | Composite ** |
| 1/15 | 0.001185 | 0.000823 | 0.002914 | 0.002992 | 0.007115 |
| 1/10 | 0.002667 | 0.001852 | 0.006556 | 0.006733 | 0.016009 |
| 1/8 | 0.004167 | 0.002894 | 0.010245 | 0.01052 | 0.025015 |
| 1/5 | 0.010667 | 0.007407 | 0.026226 | 0.026932 | 0.064037 |
| 1/2 | 0.066667 | 0.046296 | 0.163912 | 0.168323 | 0.400233 |

Notes: *The size of Pipe section is $\Phi 180\text{mm} \times 3\text{mm}$; **The size of Composite section is $h/h' = 1/10$

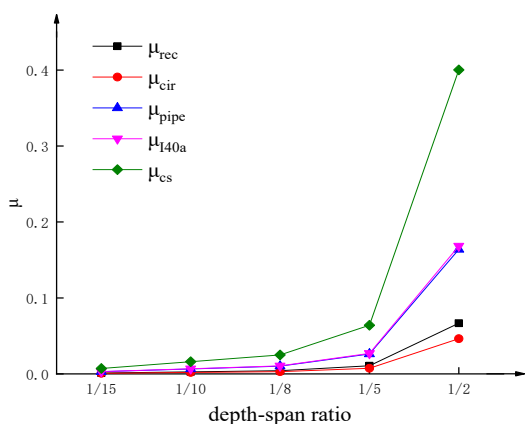


Figure 4. The change of μ with the type of cross-section and depth-span ratio.

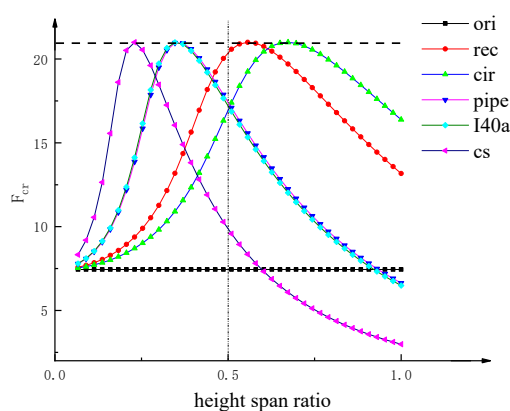


Figure 5. The change of critical under effect of associated bending moment.

As can be seen from Figure 4, μ increases with the increase of the depth-span ratio, and under the same span, $\mu_{cir} < \mu_{rec} < \mu_{pipe} < \mu_{I40a}$. In other words, the larger the section, the greater coefficient μ and the influence of shear deformation.

4.4 The effect of bending-shear accompanying term

when the associated bending-shear accompanying effect is considered, the stability equation (9) can be transformed into:

$$|K - (s^m + s^{ms})| = 0 \tag{16}$$

It is worth mentioning that equation (16) does not actually exist, because the result of F_{cr} is the combined action of bending and shear, and it cannot be separated from the pure shear term and combine with the bending term alone. To discuss the relationship between each term conveniently, equation (16) has been constructed to focus on the impact of the accompanying bending and shear term on the geometric stiffness matrix.

The stability equation is as follows:

$$\begin{vmatrix} 24 - (72 - 1440\mu)\beta & [6 - (3 - 180\mu)\beta]l & [6 - (3 - 180\mu)\beta]l \\ [6 - (3 - 180\mu)\beta]l & (8 - 4\beta)l^2 & 2l^2 \\ [6 - (3 - 180\mu)\beta]l & 2l^2 & (8 - 4\beta)l^2 \end{vmatrix} = 0 \quad (17)$$

So $F_{cr}^{ms} = \frac{7.755EI}{l^2} > F_{cr} = \frac{7.4446EI}{l^2}$, And $7.755/7.4446=4.2\%$. Therefore, the accompanying effect of bending and

shear has an increasing effect on the critical load, similar to the bending softening effect mentioned earlier, the effect of increasing the critical load can be called the accompanying stiffening effect of bending and shear.

Solve the critical loads corresponding to different section types at different depth-span ratios and organize them as shown in Figure 5.

As shown in Figure 5, it can be clearly seen that the trend of data here can be divided into three groups: The first group is the result calculated by using the unimproved geometric stiffness matrix (ori). Since s_m does not contain the μ , the result has nothing to do with the depth-span ratio and cross-section type. So it is a straight line. The second group includes rectangular section and circular section (rec, cir). With the increase of the depth-span ratio, the critical load first increases and then decreases, and there is an extreme point, and value is still larger than that without considering the coupling effect when the high-span ratio increases to 1. The third group includes circular tube section, I40a section and composite section (pipe, I40a, cs). With the increase of depth-span ratio, the critical load corresponding to these three section types also presents a form of first increase and then decrease, and there are also extreme points, but the critical load will exist the condition less than the initial value when the depth-span ratio keeps increasing.

In addition, regardless of the type, size and depth-span ratio of these sections, the extreme points corresponding to the critical load are basically the same, as shown by the black dashed line in Figure 5, but the occurrence time exists successively: composite sections appear first, followed by I40a and circular tube sections, and the circular and rectangular appear last.

If the critical load change rate k is defined as the rate at which the critical load changes with the depth-span ratio, then the relationship between the change rates: $k_{cs} > k_{I40a} \approx k_{pipe} > k_{rec} > k_{cir}$.

4.5 The effect of shear

Similarly, to reflect the influence of shear terms, the stability equation (9) can be transformed into:

$$|K - (s^m + s^s)| = 0 \quad (18)$$

The stability equation is as follows:

$$\begin{vmatrix} 24 - (72 + 8640\mu^2)\beta & [6 - (3 + 2160\mu^2)\beta]l & [6 - (3 + 2160\mu^2)\beta]l \\ [6 - (3 + 2160\mu^2)\beta]l & [8 - (4 + 1080\mu^2)\beta]l^2 & 2l^2 \\ [6 - (3 + 2160\mu^2)\beta]l & 2l^2 & [8 - (4 + 1080\mu^2)\beta]l^2 \end{vmatrix} = 0 \quad (19)$$

So $F_{ser} = \frac{7.4412EI}{l^2} < F_{cr} = \frac{7.4446EI}{l^2}$. The values of F_{cr} in different sections under different depth-span ratios can be

calculated and summarized in Figure 6.

Focusing on the influence of any section type with the change of the depth-span ratio, it can be seen that with the gradual increase of the depth-span ratio, the effect of pure shear on the reduction of the critical load of the structure is more and more obvious. By paying attention to each depth-span ratio group, it can be seen that under different section types, pure shear will reduce the critical load of the structure, and the reduction amplitude increases with the size of the section. For the composite beam with 1/5 span length, this reduction has been reduced by 22.3%, which shows that this shear softening cannot be ignored.

Since the effect of weakening is consistent with the bending softening effect proposed in 4.2 and is further generated on the basis of it, it can be called shear softening effect or softening deepening effect, and the corresponding ss is the shear

softening matrix.

4.6 The effect of compression-bending-shear accompanying term

However, when the associated compression-bending-shear accompanying term is considered, the stability equation (9) can be transformed into:

$$|K - (s^m + s^s + s^{sm} + s^{msn})| = 0 \tag{20}$$

It is noted that the improved formula (10) is related to N_e expect to μ . At this time, s^{msn} is an approximate value, and N_e is the value of critical load under unimproved conditions obtained by traditional structural mechanics calculation methods. In this question, $N_e = 7.379EI / l^2$ is substituted and the following figure is obtained after calculation:

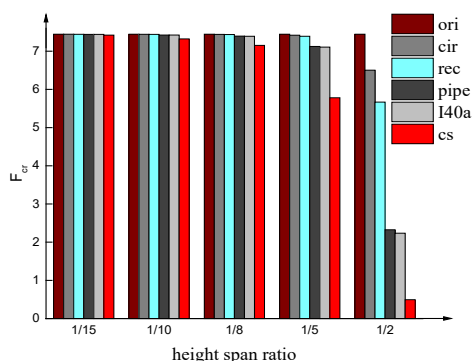


Figure 6. Change of critical load under the influence of shear deformation.

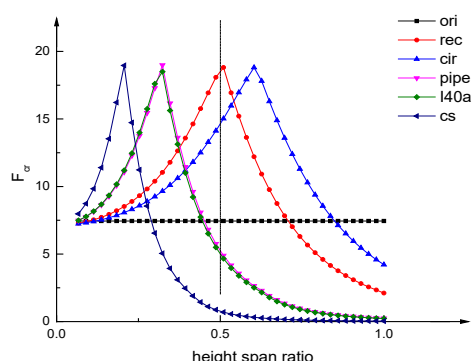


Figure 7. Change of critical load under the influence of shear deformation.

It can be seen in Figure 7 that the critical load increases first and then decreases after considering the influence of bending and shear association on the geometric stiffness matrix, and there is a equal magnitude. Although the critical load is smaller than the original load under various section types when the high-span ratio is 1, there is still a relationship between the rate of change: $k_{cs} > k_{I40a} \approx k_{pipe} > k_{rec} > k_{cir}$.

5. Conclusion

Conclusions of this paper are as follows:

- Not only μ is further discussed, but also the value table of it under common cross section types and depth-span ratio changes is given. The approximate value can be obtained by interpolation elsewhere.
- Considering that the shear term is a descending effect on the critical load, which is unfavorable to the stability of the structure, it can be called the shear softening effect, and the corresponding matrix is called the shear softening matrix; The associated terms of flexural-shear and flexural-reduction have the effect of first increasing and then decreasing the critical load, which can be called the associated hardening effect and the associated softening effect respectively, and the corresponding matrices are called the associated hardening matrix and the associated softening matrix respectively.
- The effect of the associated term on the critical load first increases and then decreases with the increase of depth-span ratio. Making full use of this improvement effect is beneficial for the design of structural stability.

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