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Some Classes of Exponentially General Variational-like Inequalities

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Abstract

In this paper, we define and introduce some new classes of exponentially variational-like inequalities, which are called exponentially general variational-like inequalities. It is shown that the optimality conditions of differentiable exponentially general preinvex functions can be characterized by exponentially general variational-like inequalities. These exponentially inequalities have applications in mathematical programming, information science, net work, data analysis and machine leaarning. We apply the auxiliary principle technique to suggest and analyse some new inertal iterative methods for finding the approximate solutions of these exponentially variational iequalities. Convergence criteria is discussed under some mild conditions. Our method of analyzing the convergence criteria is very simple as compared with other techniques. Several special casess, which can be obtained, are discussed as applications. We have only considered the theoretical aspects of these methods. Numerical implementation and comparison with other techniques is an open problem. Results obtained in this paper can be viewed as significant improvement of previously known results. The ideas and technqiues of this paper stimluate further research in different areas of pure and applied sciences.

Keywords

Convex functions, minimum, exponentially convex functions, variational-like inequalities, iterative methods

1. Introduction

Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. The concept of invex function for the differentiable functions was introduced in [1] by Hanson, which played significant part in the mathematical programming. The invex set and preinvex functions were considered in [2] by Ben-Israel and Mond. It is proved in [3] that the differentiable preinvex function are invex functions. In [3], Mohen and Neogy proved the converse also holds under certain conditions. The optimality condition for the differentiable preinvex function was characterized in [4] by a class of variational-like inequalities. More details of the properties, generalizations, extensions and applications of the preinvex functions can be found in [4-8]. We would like to emphasize that variational-like inequalities include variational inequalities and complementarily problems.

Variational inequalities can be viewed as a significant extension of the variational principles. These results have inspired a great deal of subsequent work, which has expanded the role and applications of the convexity in nonlinear optimization and engineering sciences. Some of these developments have made mutually enriching contacts with other fields. Ideas explaining these concepts had played and continue to the developments of new and powerful techniques to solve a wide class of linear and nonlinear problems.

It is known that the minimum of the differentiable convex functions can be characterized by the variational inequality and discussed its applications in the potential theory. It is amazing that this simple fact has played a significant role in the development of various fields of pure and applied sciences. There are several methods for finding the approximate solution of the variational inequalities and their variant forms. Auxiliary principle technique is quite flexible and unified one. It has shown that the principle technique include the projection method and resolvent equation technique as special cases. This technique does not require the evaluation of the projection and resolvent, which is the main advantage pf this method. The auxiliary principle technique was introduced in [9, 10] to study the existence of the solution of the mixed variational inequalities. This technique was applied in [11-13] to suggest and investigate a wide class of iterative methods for solving different classes of variational inequalities.

In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex (concave) functions, this concept and its variant forms were considered and investigated. These exponentially convex functions have important and interesting applications in data analysis, machine learning and statistics, which are explored in [14-16] and the references therein. Recently, some new concepts of exponentially convex functions are investigated. It has been shown that the exponentially convex (concave) has nice properties which convex functions enjoy. It has been shown that the exponentially convex (concave) have nice properties which convex functions enjoy.

It has been shown that the optimal conditions of the differentiable exponentially preinvex functions can be characterized by a class of exponentially variational-like inequalities. These facts motivated us to consider the exponentially general variational-like inequality, which is the main motivation of this paper. Some important cases are derived as applications of exponentially variational-like inequalities. The auxiliary principle technique is used to propose and analyze some iterative methods for solving these inequalities. Our methods of proofs are simple as compared other techniques. The ideas and techniques of this paper may be starting point for further research in these areas.

2. Preliminary Concepts

Let K be a nonempty closed set in a normed space H. We denote by $\langle .,. \rangle$ and $||.||$ the inner product and norm, respectively. For the sake of completeness and to convey the main ideas, we recall the well-known basic concepts and results.

Definition 1.

The set $K_{g\eta}$ is said to be an general invex set with respect to arbitrary bifunction $\eta(.,.)$: $H \times H \to H$, and the function g, such that

$$g(u) + t\eta(g(v), g(u)) \in K_{g\eta}, u, v \in K_{g\eta}, t \in [0,1].$$

$$For\ \eta\big(g(v), g(u)\big) = g(v) - g(u), general\ invexset K_{g\eta}\ becomes\ general\ convexset K_g.$$

We now consider the concept of the exponentially general preinvex function.

Definition 2. A function F is said to be exponentially general preinvex function, if there exists a b arbitrary bifunction $\eta(.,.)$: $H \times H \to H$, and the function g, such that

$$e^{F(g(u)+t\eta(g(v),g(u)))} \leq (1-t)e^{F(g(u))} + te^{F(g(v))}, \forall u, v \in K_{g\eta}, t \in [0,1].$$

We remark that Definition 2 can be rewritten in the following equivalent way

Definition 3. A function F is said to be exponentially general log- preinvex function, if

$$F(g(u) + t\eta(g(v), g(u))) \leq log[\,(1-t)e^{F(g(u))} + te^{F(g(v))}], \forall u, v \in K_{g\eta}, t \in [0,1].$$

A function is called the exponentially general preincave function F, if -F is exponentially general preinvex function. It is obvious that two concepts are equivalent. This equivalent has been used to discuss various aspects of the exponentially general preinvex functions.

$$For\ \eta\big(g(v), g(u)\big) = v - u, general\ invex\ set K_{g\eta}\ becomes\ convex\ set K,$$

The Definition 2 reduces to

Definition 4. A function F is said to be exponentially convex function, if

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)}, \forall u, v \in K, t \in [0,1],$$

which can be written as

$$F(u+t(v-u)) \leq log[\,(1-t)e^{F(u)} + te^{F(v)}], \forall u, v \in K, t \in [0,1].$$

It is worth mentioning that one can also deduce the concept of exponentially convex functions from r-convex functions.

For the applications of the exponentially convex functions in the mathematical programming and information theory, see, for example, in [14-17] For the applications of the exponentially concave function in the communication and information theory, we have the following example.

Example [1]: The error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

becomes an exponentially concave function in the form $erf\left(\sqrt{x}\right) and\ x \geq 0$, which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory. f the function is both preinvex and preincave, then the function is called affine preinvex function.

Definition 5. A function F is said to be exponentially affine general preinvex function, if

$$e^{F(g(u)+t\eta(g(v),g(u)))} = (1-t)e^{F(g(u))} + te^{F(g(v))}, \forall u, v \in K_{g\eta}, t \in [0,1]$$

Or equivalently

$$F(g(u)+t\eta(g(v),g(u)) = log[\,(1-t)e^{F(g(u))} + te^{F(g(v))}], \forall u, v \in K_{g\eta}, t \in [0,1]$$

In this section, we introduce a new class of variation-like inequality, which is called the exponentially general variational-like inequality. First of all, we discuss the optimality condition for the differentiable exponentially prinvex functions, which is the main motivation of our next result.

Theorem 1. Let F be a differentiable exponentially general preinvex function. Then $u \in K_{g\eta}$ is the minimum of the function F, if and only if, $u \in K_{g\eta}$ satisfies the inequality

$$\left\langle e^{F'(g(u))}, \eta(g(v), g(u)) \right\rangle \geq 0, v, u \in K_{g\eta}, \tag{1}$$

Proof: Let $u \in K_{g\eta}$ be the minimum of the differentiable general preinvex function F. Then

$$F(u) \leq F(v), \forall v \in K_{g\eta},$$

from which, we have

$$e^{F(u)} \leq e^{F(v)}, \forall v \in K_{g\eta}. \tag{2}$$

Since $K_{g\eta}$ is a general innvex set, so $\forall v, u \in K_{g\eta}, v_t = g(u) + t\eta(g(v), g(u)) \in K_{g\eta}$. Taking $v = v_t$ in (2), we have

$$0 \leq \frac{e^{F(g(u)+t\eta(g(v),g(u))} - e^{F)g(u))}}{t} = \left\langle e^{F'(g(u))}, \eta(g(v), g(u)) \right\rangle$$

the required (1).

Conversely, let F be a differentiable exponentially general preinvex function. Then

$$e^{F(g(u)+t\eta(g(v),g(u))} \leq (1-t)e^{F(g(u))} + te^{F(g(v))}, \forall u, v \in K_{g\eta}, t \in [0,1],$$

from which, using (1), we have

$$e^{F(g(v))} - e^{F(g(u))} \leq \frac{e^{F(g(u)+t\eta(g(v),g(u))} - e^{F)g(u))}}{t} = \left\langle e^{F'(g(u))}, \eta(g(v), g(u)) \right\rangle \leq 0$$

from which, we have

$$F(u) \leq F(v), \forall v \in K_{g\eta}.$$

This shows that $u \in K_{g\eta}$ is the minimum of the differentiable exponentially general preinvex function,

The inequality of the type (1) is called the exponentially variational-like inequality and appears to be new one. This fact motivated us to introduce a more general variational-like inequality of which the inequality (1) is a special case.

For a given nonlinear operator T, we consider the problem of finding $u \in K_{g\eta}$ such that

$$\langle \boldsymbol{e^{Tu}}, \boldsymbol{\eta(g(v), g(u))} \rangle \geq \mathbf{0}, \forall \boldsymbol{v} \in \boldsymbol{K_{g\eta}}, \tag{3}$$

which is called the exponentially general variational-like inequality.

3. Special Cases

We now consider some important special cases of the problem (3).

(i). Clearly for $e^{Tu} = e^{F(g(u))}F'(g(u))$, the inequality (1) is special case of inequality (3).

(ii). $If \eta(g(v), g(u)) = g(u) - g(u)$, then the problem (3) is equivalent to fining $u \in K_{g\eta}$ such that

$$\langle e^{Tu}, g(v) - g(u) \rangle \geq 0, \forall u, v \in K_{\eta}, \tag{4}$$

which is called the exponentially general variational inequality.

(iii). If $e^{Tu} = N(u)$, where $N: H \to H$ is a nonlinear operator, then problem (4) collapses to finding $u \in K_g$, such that

$$\langle Nu, g(v) - g(u) \rangle \geq 0, \forall u, v \in K_g \tag{5}$$

is called the general variational inequality.

(iv). If g= I, then problem (3) reduces to finding $u \in K_{\eta}$ such that

$$\langle e^{Tu}, \eta(v, u) \rangle \geq 0, \forall v \in K_{\eta}, \tag{6}$$

which is called the exponentially variational-like inequality studied .

(v). If $If e^{Tu} = N(u)$, and $\eta(v, u) = v - u$ then the problem (6) is equivalent to finding $u \in K$ such that

$$\langle Nu, v - u \rangle \geq 0, \forall u, v \in K,$$

which is called the classical variational inequality, introduced by Stampacchia in [13].

(vi). If $K_{g\eta}^{\bullet} = \{u \in H: \langle u, v \rangle \geq 0, \forall v \in K_{g\eta}\}$ is a dual convex cone, then problem (5) is equivalent to finding

$$g(u) \in K_{\eta}, e^{Tu} \in K_{g\eta}^{\bullet} and \ \langle N(u), g(u) \rangle = 0,$$

is called the exponentially general nonlinear complementarily problem. For the applications and numerical results, see, for example [21-23]

It is worth mentioning that for suitable and appropriate choice of the operators, convex sets and spaces, one can obtain several new and known classes of variational inequalities and complementarily problems as special cases. Consequently, it follows that the problem (3) is quite flexible and unified one.

We now define some new concepts.

Definition 6. An exponentially operator T is said to:

(i) exponentially η -monotone, if

$$\langle e^{Tu}, \eta(v, u) \rangle + \langle e^{Tv}, \eta(u, v) \rangle \leq 0, \forall u, v \in K_{\eta}.$$

(ii) quad exponentially strongly η -monotone, if there exists a constant $\xi > 0$ such that

$$\langle e^{Tu}, \eta(v, u) \rangle + \langle e^{Tv}, \eta(u, v) \rangle \leq -\xi ||\eta(v, u)||^2, \forall u, v \in K_{\eta}$$

(iii) exponentially Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$||e^{Tu} - e^{Tv}|| \leq \beta ||u - v||, \forall u, v \in K_{\eta}$$

(iv) exponentially pseudo monotone, if

$$\langle \boldsymbol{e^{Tu}}, \boldsymbol{\eta(v, u)} \rangle \geq \mathbf{0} \Rightarrow -\langle \boldsymbol{e^{Tv}}, \boldsymbol{\eta(u, v)} \rangle \geq \mathbf{0}, \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{K_{\eta}}$$

(v) An operator $T: K_{\eta} \to H$ is called exponentially η -hemicontinuous, if $\forall u, v \in K_{\eta}$, the mapping $t \in [0,1]$ implies that $\langle e^{T(u+t\eta(v,u))}, \eta(v, u) \rangle$ is continuous.

Note that every exponentially η -monotone operator is exponentially pseudo η -monotone, but the converse is not true.

We also need the following condition.

Assumption 1: The bifunction $\eta(.,.)$with respect to function g is required to satisfy

$$\boldsymbol{\eta(g(u),g(u)+t\eta(g(v),g(u))=-t\eta(g(v),g(u))}$$
$$\boldsymbol{\eta(g(v),g(u)+t\eta(g(v),g(u))=(1-t)\eta(g(v),g(u)),\forall u,v\in K_{g\eta}}$$

Note $\eta(g(v),g(u))=0\Leftrightarrow g(u)=g(v),\forall u,v\in K_{g\eta}$

Also $\eta(g(v),g(u))+\eta(g(u),g(v))=0,\forall u,v\in K_{g\eta}$. This implies that the bifunction $\eta(.,.)$is skew symmetric. For g=I, the Assumption 1 is mainly due to Mohen and Neogy [3].

Theorem 2. Let T be exponentially η -monotone operator and exponentially η -hemicontinuous.

If Assumption 1 holds and function g is linear, then $u\in K_{g\eta}$ satisfies the inequality (1), if and only if, $u\in K_{g\eta}$ satisfies

$$\langle e^{Tv},\eta(u,v\rangle\leq 0,\forall u,v\in K_{\eta} \tag{7}$$

which is known as Minty exponentially variational-like inequality.

Proof.

Let $u\in K_{g\eta}$ be a solution of the problem (1). Then $u\in K_{g\eta}$ such that

$$\langle e^{Tv},\eta(u,v\rangle\leq 0,\forall u,v\in K_{\eta},.$$

the required (7), where we used the fact that the operator T is exponentially η -monotone operator.

Conversely, since $u\in K_{g\eta}$ is a general invex set, so $v_t=g(u)+t\eta(g(v),g(u))\in K_{g\eta},u,v\in K_{g\eta}$. Taking $v=v_t$ in (7) and using the Assumption 1, we have,

$$\left\langle e^{T(g(u)+t\eta(g(v),g(u)))},\eta(g(v),g(u))\right\rangle\geq 0,\forall u,v\in K_{g\eta}.$$

Since the operator T is exponentially η -hemicontinuous, so taking the limit as $t\to 0$, we have

$$\langle e^{Tu},\eta(g(v),g(u))\rangle\geq 0,\forall v\in K_{g\eta},$$

the required result (3).

4. Results

We suggest and analyze some iterative methods for solving the exponentially general variational inequality (3) using the auxiliary principle technique. We use this technique to suggest some iterative methods for solving the problem (3).

For a given $u\in K_{g\eta}$ satisfying (3), find $u\in K_{g\eta}$ such that

$$\left\langle \rho e^{T(\varsigma w+(1-\varsigma)u)},\eta(g(v),g(w))\right\rangle+\langle w-u+\alpha(u-u),v-w,v-w\rangle\geq 0,\forall v\in K_{g\eta} \tag{8}$$

which is the auxiliary exponentially variational-like inequality, where $\alpha,\varsigma\in[0,1]$ and ρ are constants.

We note that, if $u=w$, then w is a solution of (3). This fact allows us to suggest and analyze another iterative method for solving the exponentially variational-like inequality (3).

Algorithm 1. For a given $u_0,u_1\in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\left\langle \rho e^{T(\varsigma u_{n+1}+(1-\varsigma)u_n)},\eta(g(v),g(u_{n+1}))\right\rangle+\langle u_{n+1}-u_n+\alpha_n(u_n-u_{n-1},v-u_{n+1}\rangle\geq 0,\forall v\in K_{g\eta} \tag{9}$$

Algorithm 1 is called the inertial hybrid proximal method for solving the problem (1).

For $\varsigma=1$, Algorithm1 reduces to

Algorithm 2. For a given $u_0,u_1\in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho e^{Tu_{n+1}}),\eta(g(v),g(u_{n+1}))\rangle+\langle u_{n+1}-u_n+\alpha_n(u_n-u_{n-1},v-u_{n+1}\rangle\geq 0,\forall v\in K_{g\eta} \tag{9}$$

We now study the convergence criteria of Algorithm 2 under suitable conditions.

Theorem 4. Let $u\in K_{g\eta}$ be a solution of (3) and let u_{n+1} be the approximate solution obtained from Algorithm 2. If the operator is pseudo exponentially η -monotone, then

$$\begin{aligned}||u_{n+1}-u||^2\leq{}&||u-u_n||^2-||u_{n+1}-u_n-\alpha_n(u_n-u_{n-1})||^2\\&+\alpha_n\{||u_n-u||^2-||u_n-u||^2+2||u_n-u_{n-1}||^2\}\end{aligned} \tag{10}$$

Proof. Let $u\in K_{g\eta}$ be a solution of (3). Then

$$\boldsymbol{\langle e^{Tu},\eta(g(v),g(u)\rangle\geq 0,\forall u,v\in K_{g\eta}}$$

implies that

$$-\langle \boldsymbol{e^{Tv}}, \boldsymbol{\eta(g(u), g(v))} \rangle \geq \mathbf{0}, \forall \boldsymbol{v} \in \boldsymbol{K_{g\eta}}, \tag{11}$$

where we have used that fact T is pseudo exponentially η -monotone.

Taking $v = u_{n+1}$ in (11), we have

$$-\langle \boldsymbol{e^{Tu_{n+1}}}, \boldsymbol{\eta(g(u), g(u_{n+1}))} \rangle \geq \mathbf{0}, \forall \boldsymbol{v} \in \boldsymbol{K_{g\eta}}.$$

Now taking $v = u$ in (9), we obtain

$$\langle \rho e^{Tu_{n+1}}), \eta(g(u), g(u_{n+1})) \rangle + \langle u_{n+1} - u_n + \alpha_n(u_n - u_{n-1}, u - u_{n+1} \rangle \geq 0. \tag{12}$$

From (11) and (12), we have

$$\langle u_{n+1} - u_n + \alpha_n(u_n - u_{n-1}, u - u_{n+1} \rangle \geq -\langle \rho e^{Tu_{n+1}}), \eta(g(u), g(u_{n+1})) \rangle \geq 0. \tag{13}$$

which implies that

$$\langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq +\alpha_n \langle u_n - u_{n-1}, u - u_{n+1} \rangle. \tag{14}$$

Using the inequality

$$2\langle u, v \rangle = ||u + v||^2 - ||u||^2 - ||v||^2, \forall u, v \in H$$

and rearranging the terms in (14), one can easily obtain the required result.

Theorem 5. Let H be a finite dimensional space.

Let u_{n+1} be the approximate solution obtained from Algorithm 1 and $u \in K_{g\eta}$ be a solution of (3).

If there exists $\alpha \in (0,1)$ such that $0 < \alpha_n < \alpha, \forall n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \alpha_n \, ||u_n - u_{n-1}|| \leq \infty,$$

Then $lim_{n\to\infty} u_n = u$.

Proof. Let $u \in K_{g\eta}$ be a solution of (3). First we consider the case $\alpha_n = 0$. In this case, from (10),

It follows that the sequence $\{||u_n - u_{n+1}||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from ((10), we have

$$\sum_{n=0}^{\infty} ||u_n - u_{n+1}||^2 \leq ||u - u_0||^2,$$

which implies that

$$\Sigma_{lim\to\infty} ||u_{n+1} - u_n|| = 0. \tag{15}$$

Let $\overline{u}$ be the cluster point of the sequence $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\overline{u} \in H$.

Replacing u_n by $\{u_{n_j}\}$ in (9) and taking the limit $u_j \to \infty$and using (15), we have

$$\langle e^{T\overline{u}}, \eta(g(v), g(\overline{u}) \rangle \geq 0, \forall v \in K_{g\eta}$$

which implies that $\overline{u}$ solves the exponentially general variational-like inequality problem (3) and

$$||u_{n+1} - u_n||^2 \leq ||u - u_n||^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\overline{u}$ and $lim_{n\to\infty} u_n = \overline{u}$.

Now we consider the case $\alpha_n > 0$. From (9), we have

$$\sum_{n=1}^{\infty} ||u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})||^2 \leq ||u - u_0||^2 + \sum_{n=1}^{\infty} \{\alpha_n ||u_n - u||^2 + 2||u_n - u_{n=1}||^2\} \leq \infty,$$

which implies that

$$lim_{n\to\infty} | \, ||u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})||^2 = 0.$$

Repeating the above arguments as in the case $\alpha_n = 0$, one can easily show that $lim_{n\to\infty} u_n = \overline{u}$, the required result.

We now discuss some special cases of Algorithm 1 for solving the exponentially general variational-like inequality (3).

(I). For $\alpha_n = 0$, Algorithm 1 reduces to:

Algorithm 3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\left\langle \rho e^{T(\varsigma u_{n+1}+(1-\varsigma)u_n)}, \eta(g(v), g(u_{n+1})) \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \forall v \in K_{g\eta}.$$

Algorithm 3is a hybrid proximal method for solving the problem (3).

(II). For $\varsigma = 1$ Algorithm 3 reduces to;

Algorithm 4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho e^{Tu_{n+1}}, \eta(g(v), g(u_{n+1})) \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \forall v \in K_{g\eta}.$$

Algorithm 4. is an implicit method for solving the problem (3).

(III). For $\varsigma = 1$ Algorithm 3 collapses to:

Algorithm 5. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho e^{Tu_n}, \eta(g(v), g(u_{n+1})) \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \forall v \in K_{g\eta}.$$

which is known as explicit method.

(IV). For $\varsigma = \frac{1}{2}$, Algorithm 3 collapses to:

Algorithm 6. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\left\langle \rho e^{T\left(\frac{u_{n+1}+u_n}{2}\right)}, \eta(g(v), g(u_{n+1})) \right\rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \forall v \in K_{g\eta}.$$

Algorithm 6 is called the midpoint hybrid iterative sches for solving the exponentially general variational-like inequality (3).

5. Remarks

For suitable and appropriate choice of the parameters α, ς, ρ, bifunction $\eta(.,.)$and the function g, on can obtain several new and known iterative methods for solving exponentially general variational-like inequalities and the related problems.

Using the technique of this paper, one can study the convergence of these iterative methods.

We have only convey the main idea of the exponentially variational inequalities.

6. Conclusion

In this paper, we have introduced and studied a class of convex functions, which is called the exponentially preinvex function. It has been shown that exponentially preinvex functions enjoy several properties which convex functions have. We have shown that the minimum of the differentiable exponentially preinvex functions can be characterized by a new class of variational inequalities, which is called the exponential general variational-like inequality. Some iterative methods for solving the exponentially general variational-like inequalities are proposed and analyzed using the auxiliary principle techniques. Convergence criteria of the proposed iterative schemes are analyzed under some mild weak conditions. One can explore the applications of the exponentially general variational-like inequalities. This may stimulate further research.

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