Advanced Investigation of Rocket Motion Equations through Bistatic Synthetic Aperture Radar (SAR) Techniques

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Abstract
This paper investigates the application of bistatic Synthetic Aperture Radar (SAR) technology in the analysis and optimization of rocket motion within the equatorial plane. Bistatic SAR is an advanced remote sensing technology that enables the precise measurement of the position and velocity of objects, including rockets. By incorporating bistatic SAR data, we aim to provide a comprehensive and in-depth understanding of rocket motion under various scenarios and parameters. We first study the basic motion equations of the rocket under the influence of pure gravity and Earth’s rotation, introducing the concept of Coriolis force. Subsequently, we conduct a detailed analysis of the rocket’s dynamics, including factors such as thrust, lift, and drag, as well as their impact on the rocket at different stages. We also discuss the motion differential equations during the rocket separation process, focusing on the dynamic characteristics of the Musk-style rocket during the first stage separation and return landing. Throughout the derivation process, we employ multiple assumptions and parameters, making the rocket’s motion differential equations more complex and thus closer to actual situations. The integration of bistatic SAR technology with the rocket’s motion analysis provides a novel approach to improving rocket performance and navigation capabilities.

Keywords
Rocket motion analysis, Coriolis force, Stage separation dynamics, Bistatic Synthetic Aperture Radar

1. Methodology
In this section, we present the methodology for analyzing rocket motion [1] within the equatorial plane, taking into account various forces that influence the rocket’s trajectory, such as gravity, Earth’s rotation, and aerodynamics. The ultimate goal is to provide a solid foundation for the integration of bistatic SAR [2] technology in later stages of the study, which will further enhance our understanding of rocket dynamics and improve the design and control strategies.

We consider the rocket’s motion within the equatorial plane. The following variables are given:

- \( t \): Time,
- \( \Omega \): Earth’s rotational angular velocity,
- \( r \): Distance from the rocket to the Earth’s center,
- \( \theta \): Angle from the rocket to the launch point,
- \( r_0 \): Earth’s radius,
- \( g \): Gravity acceleration on the ground (excluding centrifugal force from Earth’s rotation),
- \( R \): Radial component of the thrust per unit mass received by the rocket on average,
- \( \Theta \): Tangential component of the thrust per unit mass received by the rocket on average.

We can use Newton’s second law to describe the dynamics of radial and tangential components.
For the radial component, the net force on the rocket includes gravity, centrifugal force, and the radial component of thrust. By Newton’s second law:

\[ m \frac{d^2 r}{dt^2} = -mg \left( \frac{r_0}{r} \right)^2 + mr \left( \dot{\theta} \pm \Omega \right)^2 + mR \]

Where \( m \) is the rocket’s mass. Simplifying the equation, we get:

\[ \frac{d^2 r}{dt^2} = -g \left( \frac{r_0}{r} \right)^2 + r \left( \dot{\theta} \pm \Omega \right)^2 + R \]

For the tangential component, the net force on the rocket is the tangential component of thrust and the Coriolis force. The expression for Coriolis force is:

\[ F_{cor} = -2m v_{rel} \times \Omega \]

Here, \( F_{cor} \) is the Coriolis force, \( m \) is the object’s mass, \( v_{rel} \) is the object’s velocity relative to the Earth’s rotation, and \( \Omega \) is the angular velocity vector of the Earth’s rotation.

The rocket moves within the equatorial plane, and its velocity vector has radial and tangential components without a vertical component. We can represent the rocket’s velocity vector as:

\[ v_{rel} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} \]

Here, \( \hat{r} \) and \( \hat{\theta} \) are the radial and tangential unit vectors, respectively.

Next, we calculate the influence of the Coriolis force on the tangential component. Since \( v_{rel} \) and \( \Omega \) are orthogonal, we can simplify the Coriolis force formula to:

\[ F_{cor,\theta} = -2m (v_{rel} \times \Omega) \cdot \hat{\theta} \]

Now, we need to compute the vector product \( v_{rel} \times \Omega \). The definition of the vector product is:

\[ A \times B = ||A|| ||B|| \sin \alpha \hat{n} \]

Here, \( ||A|| \) and \( ||B|| \) are the lengths of vectors \( A \) and \( B \), respectively, \( \alpha \) is the angle between them, and \( \hat{n} \) is the unit vector perpendicular to \( A \) and \( B \).

For \( v_{rel} \times \Omega \), we have:

\[ v_{rel} \times \Omega = (\dot{r} \hat{r} + r \dot{\theta} \hat{\theta}) \times (\Omega \hat{k}) \]

Since \( \hat{r} \), \( \hat{\theta} \), and \( \hat{k} \) are mutually orthogonal unit vectors, we can compute the components of the vector product. The tangential component of the first vector product:

\[ (\dot{r} \hat{r} \times \Omega \hat{k}) \cdot \hat{\theta} = -\dot{r} \Omega \]

The tangential component of the second vector product:

\[ (r \dot{\theta} \hat{\theta} \times \Omega \hat{k}) \cdot \hat{\theta} = 0 \]

The second vector product is 0 because \( \hat{\theta} \) and \( \hat{k} \) are orthogonal.

Now we add these two vector products, obtaining:

\[ (v_{rel} \times \Omega) \cdot \hat{\theta} = -\dot{r} \Omega \]

Next, we substitute the result into the formula for the tangential component of the Coriolis force:

\[ F_{cor,\theta} = -2m (v_{rel} \times \Omega) \cdot \hat{\theta} = -2m(-\dot{r} \Omega) = 2m \dot{r} \Omega \]

Thus, the influence of the Coriolis force on the tangential component is:

\[ F_{cor,\theta} = 2m \dot{r} \left( \dot{\theta} \pm \Omega \right) \]

For the tangential direction, the expression of Coriolis force is:

\[ -2m \dot{r} \left( \dot{\theta} \pm \Omega \right) \]

Hence, Newton’s second law in the tangential direction is:

\[ mR \frac{d^2 \theta}{dt^2} = m\theta - 2m \dot{r} \left( \dot{\theta} \pm \Omega \right) \]

For an object moving within the equatorial plane, the velocity vector \( \vec{v} \) also lies within the equatorial plane. Therefore, the result of the Coriolis force \( 2m \vec{v} \times \vec{\Omega} \) will also be within the equatorial plane. That is, within the equatorial plane, the Coriolis force does not change the object’s trajectory, causing it to deviate from the plane.

To use numerical methods, we need to convert these two second-order equations into a set of first-order equations.
We can introduce new variables \( v_r = \frac{dr}{dt} \) and \( v_\theta = r \frac{d\theta}{dt} \). Then, we can rewrite the original equations as the following set of first-order equations: [4]

\[
\begin{align*}
\frac{dr}{dt} &= v_r \\
\frac{d\theta}{dt} &= v_\theta \\
\frac{dv_r}{dt} &= -g \left( \frac{r_0}{r} \right)^2 + r(\dot{\theta} \pm \Omega)^2 + R \\
r \frac{dv_\theta}{dt} &= \theta - 2v_r(\dot{\theta} \pm \Omega)
\end{align*}
\]

Now we have a set of first-order ordinary differential equations. To solve this set of equations, we need initial conditions, such as initial time, initial position, initial velocity, etc.

Next, we will discuss how to optimize rocket design and control strategies using bistatic Synthetic Aperture Radar (SAR) technology. First, let us assume that the bistatic SAR system can measure the position and velocity information of the rocket at different stages. We can use this information to estimate the dynamic parameters of the rocket, thus optimizing rocket design and control strategies.

Let \( P(t) = (x(t), y(t), z(t)) \) be the spatial position of the rocket at a certain moment, and \( V(t) = (V_x(t), V_y(t), V_z(t)) \) be the velocity of the rocket at that moment. Using bistatic SAR technology, we can measure \( P(t) \) and \( V(t) \) and obtain more accurate estimates through data fusion techniques.

Next, we combine the measured position and velocity information with the rocket motion differential equations to optimize rocket design and control strategies. First, we can calculate the air resistance and Earth's gravity effects on the rocket at different stages. Let \( F_D(t) \) be the air resistance experienced by the rocket at a certain moment, and \( F_G(t) \) be the Earth's gravity effect on the rocket at that moment. We can calculate them using the following formulas:

\[
F_D(t) = \frac{1}{2} \rho(P(t)) |V(t)|^2 C_D A \\
F_G(t) = m(t) g \left( \frac{r_0}{|P(t)|} \right)^2
\]

Here, \( C_D \) is the drag coefficient of the rocket, \( A \) is the cross-sectional area of the rocket, and \( \rho(P(t)) \) is the air density at the position \( P(t) \).

Then, based on the air resistance and Earth’s gravity effects, we can adjust the rocket’s attitude angles and thrust to achieve optimized rocket design and control strategies. For example, we can reduce the impact of air resistance and improve the rocket’s operational efficiency by adjusting the rocket’s attitude angles \( \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \gamma_1(t), \gamma_2(t), \) and \( \gamma'_1(t) \).

Moreover, we can utilize bistatic SAR technology to predict the rocket’s future motion trajectory and optimize its control strategies accordingly. For instance, by solving the rocket motion differential equations, we can predict the rocket’s position and velocity information at a future moment, and then adjust the rocket’s thrust and attitude angles to achieve more accurate navigation [5].

To numerically solve the first-order ordinary differential equations, we can employ numerical integration methods, such as the Runge-Kutta method, to calculate the rocket’s motion trajectory step by step, using the initial conditions and the measured position and velocity information from the bistatic SAR system.

Under the assistance of bistatic SAR, we can further explore the deep implications of some formulas in the context of rocket design and control strategies. One way to do this is by using the Kalman filter to combine the measurements from bistatic SAR with the rocket motion model to estimate the rocket’s state more accurately.

Let’s assume that the state vector of the rocket is given by \( X(t) = [P(t), V(t)]^T \), where \( P(t) \) and \( V(t) \) are the position and velocity vectors, respectively. The rocket’s motion model can be described by a set of first-order ordinary differential equations, as discussed earlier.

The state-transition matrix, \( F(t) \), can be approximated by the Jacobian matrix of the rocket’s motion model with respect to the state vector \( X(t) \), and the control-input matrix, \( B(t) \), can be approximated by the Jacobian matrix of the rocket’s motion model with respect to the control inputs, such as thrust and attitude angles [6].

Now, let’s assume that the bistatic SAR measurements are given by \( Z(t) = [P'(t), V'(t)]^T \), where \( P'(t) \) and \( V'(t) \) are the measured position and velocity vectors, respectively. The measurement matrix, \( H(t) \), maps the true state \( X(t) \) to the measured state \( Z(t) \), and the measurement noise covariance matrix, \( R(t) \), represents the uncertainties in the bistatic SAR measurements.
The Kalman filter algorithm can be summarized as follows:

Predict the state vector and state covariance matrix for the next time step:

$$
\dot{X}_{k+1} = F_k \dot{X}_k + B_k U_k \\
\dot{P}_{k+1} = F_k P_k F_k^T + Q_k
$$

where $\dot{X}_k$ and $P_k$ are the predicted state vector and state covariance matrix at time step $k$, $F_k$ and $B_k$ are the state-transition and control-input matrices at time step $k$, $U_k$ is the control input vector at time step $k$, and $Q_k$ is the process noise covariance matrix at time step $k$.

Update the state vector and state covariance matrix using the bistatic SAR measurements:

$$
K_k = P_k H_k^T (H_k P_k H_k^T + R_k)^{-1}
$$

$$
\dot{X}_{k+1} = \dot{X}_k + K_k (Z_{k+1} - H_{k+1} \dot{X}_{k+1})
$$

$$
\dot{P}_{k+1} = (I - K_k H_{k+1}) P_k
$$

where $K_k$ is the Kalman gain matrix at time step $k$, $H_{k+1}$ is the measurement noise covariance matrix at time step $k$, and $R_k$ is the measurement noise covariance matrix at time step $k$.

The Kalman filter provides an optimal estimate of the rocket’s state by fusing the bistatic SAR measurements with the rocket’s motion model. With more accurate state estimates, we can further optimize the rocket’s design and control strategies.

For example, based on the updated state estimates, we can adjust the rocket’s thrust and attitude angles to minimize fuel consumption or maximize the mission objectives. Let $J$ be the cost function to be minimized, which can be defined in terms of fuel consumption, mission objectives, or a combination of both. The optimization problem can be formulated as:

$$
\min_{U(t)} J(\dot{X}(t), U(t))
$$

subject to the rocket’s motion equations and constraints on control inputs:

$$
d\dot{X}(t) = F(\dot{X}(t), U(t))
$$

$$
U_{min} \leq U(t) \leq U_{max}
$$

where $U(t)$ is the control input vector, $U_{min}$ and $U_{max}$ are the lower and upper bounds of the control inputs, and $F(\dot{X}(t), U(t))$ is the rocket’s motion model.

After introducing these additional parameters and assumptions, the rocket’s differential equations become more complex:

$$
m(t) \frac{d^2 r}{dt^2} = -mg(t) \left(\frac{r_0}{r}\right)^2 + r(\dot{\theta} \pm \Omega)^2 + S(t) \cos(q(t)) - \frac{1}{2} \rho(r)V^2(t)C_D A
$$

$$
m(t) \frac{d^2 \theta}{dt^2} = S(t) \sin(q(t)) \cos(q(t)) - \frac{1}{2} \rho(r)V^2(t)C_D A - 2\dot{r}(\dot{\theta} \pm \Omega) + m(t) \gamma(t)
$$

Here, $C_D$ is the drag coefficient, $C_L$ is the lift coefficient, and $A$ is the rocket’s reference cross-sectional area.

It should be noted that these equations are already quite complex and may be difficult to solve. In the actual analysis and design process, numerical methods are usually needed to solve these complex differential equations. Moreover, these equations may need further adjustment and refinement based on the rocket’s specific design and flight conditions.

When considering a situation like Space X’s Falcon 9 rocket, where a part of the rocket (the first stage) returns safely after separation, we need to split the rocket’s motion differential equations into two parts, describing the motion of the first and second stages separately. Additionally, the rocket’s mass, thrust, air resistance, and other factors will change over time, requiring the use of time-dependent parameters.

First, let’s consider the stage from the launch to the separation of the first stage rocket. Let $m_1(t)$ be the mass of the first stage rocket, $S_1(t)$ be the thrust of the first stage rocket, $r_1(t)$ be the altitude of the first stage rocket, and $\theta_1(t)$ be the azimuth angle of the first stage rocket. In this case, the motion differential equations of the first stage rocket can be expressed as:

$$
m_1(t) \frac{d^2 r_1}{dt^2} = -m_1(t) g \left(\frac{r_0}{r_1}\right)^2 + r_1(\dot{\theta} \pm \Omega)^2 + S_1(t) \cos(q_1(t)) - \frac{1}{2} \rho(r_1)V_1^2(t)C_D A_1
$$

$$
m_1(t) r_1 \frac{d^2 \theta_1}{dt^2} = S_1(t) \sin(q_1(t)) \cos(q_1(t)) - \frac{1}{2} \rho(r_1)V_1^2(t)C_D A - 2\dot{r}_1(\dot{\theta} \pm \Omega) + m_1(t) \gamma_1(t)
$$
Next, let’s consider the stage from the separation of the second stage rocket to the launch of the target orbit. Let $m_2(t)$ be the mass of the second stage rocket, $S_2(t)$ be the thrust of the second stage rocket, $r_2(t)$ be the altitude of the second stage rocket, and $\beta_2(t)$ be the azimuth angle of the second stage rocket. In this case, the motion differential equations of the second stage rocket can be expressed as:

$$
\begin{align*}
    m_2(t) \frac{d^2 \mathbf{r}_2}{dt^2} &= -m_2(t)g \left( \frac{\mathbf{r}_2}{r_2^2} \right)^2 + r_2(\dot{\theta}_2 \pm \Omega)^2 + S_2(t) \cos \alpha_2(t) - \frac{1}{2} \rho(r_2)V_2^2(t)CD2A_2 \\
    m_2(t)r_2 \frac{d^2 \theta_2}{dt^2} &= S_2(t) \sin \alpha_2(t) \cos \beta_2(t) - \frac{1}{2} \rho(r_2)V_2^2(t)C_LA_2 - 2r_2(\dot{\theta}_2 \pm \Omega) + m_2(t)r_2 \dot{\gamma}_2(t)
\end{align*}
$$

Finally, let’s consider the stage from the separation of the first stage rocket to its return and landing. In this stage, the thrust $S_1(t)$ of the first stage rocket will decrease continuously until it is shut down. The rocket will be affected by air resistance and gravity. Let $r_1'(t)$ be the altitude of the first stage rocket during the return process, and $\beta_1'(t)$ be the azimuth angle of the first stage rocket during the return process. In this stage, the motion differential equations of the first stage rocket returning can be expressed as:

$$
\begin{align*}
    m_1(t) \frac{d^2 \mathbf{r}_1'}{dt^2} &= -m_1(t)g \left( \frac{\mathbf{r}_1'}{r_1'^2} \right)^2 + r_1'(\dot{\theta}_1' \pm \Omega)^2 - \frac{1}{2} \rho(r_1')V_1'^2(t)CD1A_1 \\
    m_1(t)r_1' \frac{d^2 \theta_1'}{dt^2} &= -\frac{1}{2} \rho(r_1')V_1'^2(t)C_LA_1 - 2r_1'(\dot{\theta}_1' \pm \Omega) + m_1(t)r_1' \dot{\gamma}_1'(t)
\end{align*}
$$

Here, $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$, $\gamma_1(t)$, $\gamma_2(t)$, and $\gamma_1'(t)$ represent the attitude angles of the first and second stage rockets as well as the first stage rocket during the return process. These angles can be adjusted by the control system to meet the navigation requirements of the rocket.

These equations describe the motion of the first and second stages of the rocket and the return process of the first stage rocket. It should be noted that many parameters in these equations (such as mass, thrust, velocity, etc.) change with time, so they need to be adjusted according to the actual situation in practical applications. Also, when solving these equations in practice, numerical methods need to be used for step-by-step integration.

Next, we can explore how to use bistatic SAR technology to optimize rocket design and control strategies. First, we assume that the bistatic SAR system can measure the position and velocity information of the rocket at different stages. We can use this information to estimate the dynamic parameters of the rocket, thereby optimizing rocket design and control strategies.

Let $P(t) = (x(t), y(t), z(t))$ be the spatial position of the rocket at a certain moment, and $V(t) = (V_x(t), V_y(t), V_z(t))$ be the velocity of the rocket at that moment. Using bistatic SAR technology, we can measure $P(t)$ and $V(t)$ and obtain more accurate estimates through data fusion techniques.

Next, we will combine the measured position and velocity information with the differential equations of motion to optimize rocket design and control strategies. First, we can calculate the air resistance and Earth’s gravitational influence on the rocket at different stages. Let $F_D(t)$ be the air resistance on the rocket at a certain moment, and $F_G(t)$ be the Earth’s gravitational influence on the rocket at that moment. We can calculate them using the following formulas:

$$
\begin{align*}
    F_D(t) &= \frac{1}{2} \rho(P(t)|V(t)|^2 CD A \\
    F_G(t) &= m(t)g \left( \frac{r_0}{|P(t)|} \right)^2
\end{align*}
$$

Where $C_D$ is the drag coefficient of the rocket, $A$ is the cross-sectional area of the rocket, and $\rho(P(t))$ is the air density at the position $P(t)$.

Next, we can adjust the rocket’s attitude angles and thrust based on the air resistance and Earth’s gravitational influence to achieve optimized rocket design and control strategies. For example, we can reduce the impact of air resistance and improve rocket operation efficiency by adjusting the rocket’s attitude angles $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$, $\gamma_1(t)$, $\gamma_2(t)$, and $\gamma_1'(t)$.

Moreover, we can use bistatic SAR technology to predict the rocket’s future motion trajectory and optimize the rocket’s control strategy accordingly [7]. For instance, we can predict the rocket’s position and velocity information at a future moment by solving the rocket motion’s differential equations, and then adjust the rocket’s thrust and attitude angles to achieve more accurate navigation and reduce fuel consumption [8]. Specifically, we can solve the rocket motion’s differential equations [9] at each time step and adjust the rocket’s control strategy based on the prediction results [10].
2. Conclusion

In summary, this article provides a comprehensive and highly quantitative rocket motion differential equation analysis framework, incorporating bistatic SAR technology to improve the understanding of the dynamic behavior of rockets at various stages. Our analysis shows that the motion of rockets is influenced by multiple factors, such as the gravitational field, Coriolis force, aerodynamics, and the valuable input from bistatic SAR measurements. By deeply studying these factors and their interactions, we can better understand the motion characteristics of rockets, thus optimizing rocket design and control strategies using advanced estimation and optimization techniques.

The rocket motion differential equations we derived are only theoretical models, and they need to be adjusted according to specific situations in practical applications. Nevertheless, this theoretical framework, enriched by the integration of bistatic SAR technology, has high reference value. It provides rocket scientists and engineers with a systematic method to analyze rocket dynamic characteristics, enabling more accurate state estimation and control adjustments, which in turn help promote the continuous progress of rocket technology.

References