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An Inequality with Doubling Measure in (quasi-) Banach Function Spaces

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Abstract

Let $Z = \{Z_Q\}$ be a family of normalized (quasi-) Banach function spaces and μ a doubling measure. In Section 1, we give a brief introduction of this paper. In Section 2, we introduce some basic definitions about this article and extend the definitions of some maximal functions and some notions about them with Lebesgue measure to the case of generalized measure μ in R^n. In Section 3, we consider the inequality in [1,Theorem1.1] with doubling measure and in (quasi-) Banach function spaces. We find that the properties of Banach function spaces and the $A_\infty(d\mu)$ condition are used to get the conclusion, which are just the same as the conditions of the estimate of the $BMO(d\mu)$ functions in [2,Theorem1.1]. Particularly, the $A_\infty(d\mu)$ condition could be seen as a separation condition in Banach function spaces after using Calderon-Zygmund decomposition and may be available to solve other similar problems. Then we apply it to the localized Lorentz spaces with $0 < p < q < \infty$. This section is the main part of the article. In section4, we focused on the application of the Variable exponent L^p-spaces.

Keywords

Banach function spaces, Doubling measure, Lorentz spaces, Variable exponent L^p-spaces

1. Introduction

Javier Canto and Carlos Perez proved an inequality in the space of $L^p\left(Q, \frac{\omega dx}{\omega_r(Q)}\right)$ in [1,Theorem1.1], we rephrased the theorem as Lemma 1.1 with some abbreviation.

Lemma 1.1 Let f be a locally integrable function and ω a non-negative weight. Then for any cube Q, for any $1 \le p < \infty$ and $1 < r < \infty$, the following estimate holds

$$\left(\frac{1}{\omega_r(Q)}\int_Q \left(\frac{M_Q(f - f_Q)(x)}{M^\# f(x)}\right)^p \omega(x)\,dx\right)^{\frac{1}{p}} \le c_n p r'$$

In this article,we consider it in the case of doubling measure without any weight in Theorem 3.1. Combing with

Lemma 3.2,we obtain a simple application in Corollary 3.3.

After that,using the same method,we extend it to the (quasi-)Banach function spaces in Theorem 3.4 and find out it could be used to estimate the localized Lorentz quasi-norm of the corresponding function in remark 3.6.

Eventually,we apply the result in (quasi-) Banach function spaces to the Variable exponent L^p -spaces in Theorem 4.5 after verifying some conditions in Lemma 4.1,4.2,4.3,4.4.

1. Preliminaries

2.1 Doubling measure

Definition 2.1 A measure μ in R^n is called doubling measure, or said to satisfy the doubling condition, if there exist some positive constants c_μ and n_μ such that,for every pair of cubes E and Q in R^n with $E \subset Q$, the inequality

$$\mu(Q) \le c_\mu \left(\frac{l(Q)}{l(E)} \right)^{n_\mu} \mu(E)$$

holds, where $c_\mu > 1$.

Remark 2.2 The doubling measure considered in this article will be nontrivial to ensure that $\mu(Q) > 0$ for every cube Q in R^n (see the proof of [2,Lemma A.2]).

2.2 Maximal functions

Now we introduce some well known maximal functions with measure μ in R^n and some notations with respect to them.

Definition 2.3

$$M_\mu^{\#} h(x) = \sup_{x \in R} \frac{1}{\mu(R)} \int_R \left| h(y) - h_{R,\mu} \right| d\mu(y)$$

$$M_{Q,\mu} h(x) = \sup_{\substack{R \in D(Q) \\ x \in R}} \frac{1}{\mu(R)} \int_R |h| \, d\mu$$

$$osc_\mu(f, Q) = \frac{1}{\mu(Q)} \int_Q \left| f - f_{Q,\mu} \right| d\mu$$

$$F_{Q,\mu}(x) = \frac{\left| f(x) - f_{Q,\mu} \right|}{osc_\mu(f, Q)}$$

where R,Q are cubes in R^n, f,h are locally integrable functions, $D(Q)$ denotes the set of all dyadic cubes with respect to Q, $h_{R,\mu} = \frac{1}{\mu(R)} \int_R h(y) \, d\mu(y)$.

2.3 Banach function spaces

The following definitions are just the same as in [2].

Definition 2.4 Let X be a vector space. A function $\| \cdot \| : X \to [0, \infty)$ is called a quasi-norm if there is a constant $K \ge 1$ such that

(1) $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \|x\|$ for all $\alpha \in R$ and $x \in X$.

(3) $\|x_1 + x_2\| \le K\left(\|x_1\| + \|x_2\|\right)$ for all $x_1, x_2 \in X$.

The constant K will be called the geometric constant of $\| \cdot \|$.

A quasi-norm $\| \cdot \|$ over a vector space X will be denoted by $\| \cdot \|_X$. In case $K = 1$, the term "quasi" for the notation will be skipped.

Definition 2.5 The $A_\infty(d\mu)$ condition for families of Banach spaces

Let μ be a measure in R^n. A family of Banach spaces $Z=\{Z_Q\}$ or quasi-Banach spaces with triangle inequality constant uniformly bounded will be said to satisfy an $A_\infty(d\mu)$ condition if there exist some constant $C_Z>0$ and some increasing bijection $\psi:[0,1]\to[0,1]$ such that

$$\left\|\sum_j h_j \chi_{Q_j}\right\|_{Z_Q} \le C_Z \psi^{-1}\left[\frac{\mu\left(\underset{j}{\cup} Q_j\right)}{\mu(Q)}\right]$$

For every $\{Q_j\}\in\Delta(Q)$ and every family of functions $\{h_j\}$ satisfying $\|h_j\|_{Z_{Q_j(d\mu)}}=1$ for every $j\in N$.

Where $\Delta(Q)$ is the family of countable disjoint families of subcubes of a given cube Q.

Definition2.6 A family of (quasi-)Banach spaces $Z=\{Z_Q\}$ with triangle inequality constant uniformly bounded. μ is a measure in R^n. The family Z will be said to be good if:

(1) (Fatou property) If $\{f_k\}$ are functions in $Z_Q(d\mu)$ with $|f_k|\uparrow|f|$ $\mu-a.e.$ then $\|f_k\|_{Z_Q}\uparrow\|f\|_{Z_Q}$.

(2) (Average property) $\|\chi_Q\|_{Z_Q}\le 1$ for every cube Q in R^n.

Variable exponent L^p-spaces with measure μ in R^n

Definition 2.7 Let $p:R^n\to[1,\infty)$ be a Lebesgue measurable function and denote $p^+=ess\sup p(x)=\inf\{\alpha>0:\mu(\{x\in R^n:p(x)>\alpha\})=0\}$. Assume $p^+<\infty$. The Lebesgue space with variable exponent $p(\cdot)$ with a measure μ in R^n is the space of μ-measurable functions f satisfying that

$$\|f\|_{L^{p(\cdot)}(d\mu)}:=\inf\left\{\delta>0:\int_{R^n}\left(\frac{|f(x)|}{\delta}\right)^{p(x)}d\mu(x)\le 1\right\}<\infty$$

One may associate to this space the local averages

$$\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}:=\inf\left\{\delta>0:\frac{1}{\mu(Q)}\int_Q\left(\frac{|f(x)|}{\delta}\right)^{p(x)}d\mu(x)\le 1\right\}<\infty$$

3. Main result

In this section we will present and prove our main result.

Theorem 3.1 Let f be a locally integrable function and μ a doubling measure in R^n. Then for any cube Q and any $1\le p<\infty$, the following inequality holds

$$\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{M_{Q,\mu}(f-f_{Q,\mu})(x)}{M_\mu^\# f(x)}\right)^p d\mu(x)\right)^{\frac{1}{p}}\le c_\mu 2^{n_\mu} e(p+1)$$

Proof. Fix a cube Q, and consider the functions in Definition 2.3, we can get that, for each $x\in Q$,

$$M_\mu^\# f(x)=\sup_{x\in R}\frac{1}{\mu(R)}\int_R|f-f_{R,\mu}|d\mu\ge\frac{1}{\mu(Q)}\int_Q|f-f_{Q,\mu}|d\mu=o\ c_\mu\{f,Q)$$

As $\frac{1}{\mu(Q)}\int_Q F_{Q,\mu}(x)d\mu(x)=\frac{1}{\mu(Q)}\int_Q \frac{\left|f(x)-f_{Q,\mu}\right|}{osc_\mu(f,Q)}d\mu(x)=1$

We can apply the local Calderon-Zygmund decomposition to $F_{Q,\mu}$ in Q at level $\lambda>1$. Then there is a family of dyadic pairwise disjoint subcubes $\{Q_j\}$ with respect to Q satisfying that

(1) $\lambda<\frac{1}{\mu(Q_j)}\int_{Q_j} F_{Q,\mu}d\mu=\frac{1}{\mu(Q_j)}\int_{Q_j}\frac{\left|f-f_{Q,\mu}\right|}{osc_\mu(f,Q)}d\mu\le c_\mu 2^{n_\mu}\lambda$

(2) $\sum_j \mu(Q_j)\le\frac{1}{\lambda}\mu(Q)$

(3) For $\mu-a.e.\ x\notin\bigcup_j Q_j$, $F_{Q,\mu}(x)=\frac{\left|f(x)-f_{Q,\mu}\right|}{osc_\mu(f,Q)}\le\lambda$

Since for every $x\notin\bigcup_j Q_j$,

$$\frac{1}{\mu(R)}\int_R F_{Q,\mu}d\mu=\frac{\frac{1}{\mu(R)}\int_R\left|f-f_{Q,\mu}\right|d\mu}{osc_\mu(f,Q)}\le\lambda$$

For all dyadic R that contains x.
Therefore, for each $x\notin\bigcup_j Q_j$,

$$M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)=\sup_{\substack{R\in D(Q)\\ x\in R}}\frac{1}{\mu(R)}\int_R\left|f-f_{Q,\mu}\right|d\mu\le\lambda osc_\mu(f,Q)$$

And thus, $\frac{M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)}{M_\mu^\# f(x)}\le\frac{M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)}{osc_\mu(f,Q)}\le\lambda$.

For $x\in Q_j$, as Q_j is the maximal dyadic cube R that contains x satisfying

$\frac{1}{\mu(R)}\int_R F_{Q,\mu}d\mu>\lambda$ or

$$\frac{1}{\mu(R)}\int_R\left|f-f_{Q,\mu}\right|d\mu>\lambda osc_\mu(f,Q)$$

Hence,

$$M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)=\sup_{\substack{R\in D(Q)\\ x\in R}}\frac{1}{\mu(R)}\int_R\left|f-f_{Q,\mu}\right|d\mu=\sup_{\substack{R\in D(Q_j)\\ x\in R}}\frac{1}{\mu(R)}\int_R\left|f-f_{R,\mu}\right|d\mu=M_{Q_j}\left(f-f_Q\right)(x)$$

$$\le M_{Q_j,\mu}\left(f-f_{Q_j,\mu}\right)(x)+\left|f_{Q_j,\mu}-f_{Q,\mu}\right|$$

Furthermore, for any $x\in Q$, and thus for any $x\in Q_j$, we have

$$\frac{\left|f_{Q_j,\mu}-f_{Q,\mu}\right|}{M_\mu^\# f(x)}=\frac{\left|\frac{1}{\mu(Q_j)}\int_{Q_j}\left(f-f_{Q,\mu}\right)d\mu\right|}{M_\mu^\# f(x)}\le\frac{\frac{1}{\mu(Q_j)}\int_{Q_j}\left|f-f_{Q,\mu}\right|d\mu}{M_\mu^\# f(x)}\le\frac{\frac{1}{\mu(Q_j)}\int_{Q_j}\left|f-f_{Q,\mu}\right|d\mu}{osc_\mu(f,Q)}\le c_\mu 2^{n_\mu}\lambda$$

Therefore, for every $x\in Q$,

$$\frac{M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)}{M_\mu^\# f(x)}=\frac{M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)}{M_\mu^\# f(x)}\chi_{Q\setminus\bigcup_j Q_j}(x)+\sum_j\frac{M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)}{M_\mu^\# f(x)}\chi_{Q_j}(x)$$

$$\le \lambda \chi_{Q\setminus \cup_j Q_j}(x) + \sum_j \left(\frac{M_{Q_j,\mu}\left(f - f_{Q_j,\mu}\right)(x)}{M_\mu^\# f(x)} + \frac{\left| f_{Q_j,\mu} - f_{Q,\mu} \right|}{M_\mu^\# f(x)} \right) \chi_{Q_j}(x)$$

$$\le \lambda \chi_{Q\setminus \cup_j Q_j}(x) + \sum_j \frac{M_{Q_j,\mu}\left(f - f_{Q_j,\mu}\right)(x)}{M_\mu^\# f(x)} \chi_{Q_j}(x) + c_\mu 2^{n_\mu} \lambda \chi_{\cup_j Q_j}(x)$$

$$\le c_\mu 2^{n_\mu} \lambda + \sum_j \frac{M_{Q_j,\mu}\left(f - f_{Q_j,\mu}\right)(x)}{M_\mu^\# f(x)} \chi_{Q_j}(x)$$

For convenience,we write

$G_{Q,\mu}(x) = \dfrac{M_{Q,\mu}\left(f - f_{Q,\mu}\right)(x)}{M_\mu^\# f(x)}$ in short.

Then set $X = \sup_Q \left(\frac{1}{\mu(Q)} \int_Q G_{Q,\mu}(x)^p \, d\mu(x) \right)^{\frac{1}{p}}$, triangle inequality and disjointness of $\{Q_j\}$ yields that

$$\left(\frac{1}{\mu(Q)} \int_Q G_{Q,\mu}(x)^p d\mu(x) \right)^{\frac{1}{p}} \le c_\mu 2^{n_\mu} \lambda + \left(\frac{1}{\mu(Q)} \int_Q \left(\sum_j G_{Q_j,\mu}(x) \chi_{Q_j}(x) \right)^p d\mu(x) \right)^{\frac{1}{p}}$$

$$= c_\mu 2^{n_\mu} \lambda + \left(\frac{1}{\mu(Q)} \sum_j \int_{Q_j} G_{Q_j,\mu}(x)^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$$= c_\mu 2^{n_\mu} \lambda + \left(\sum_j \frac{\mu(Q_j)}{\mu(Q)} \frac{1}{\mu(Q_j)} \int_{Q_j} G_{Q_j,\mu}(x)^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$$\le c_\mu 2^{n_\mu} \lambda + X \left(\frac{\sum_j \mu(Q_j)}{\mu(Q)} \right)^{\frac{1}{p}} \le c_\mu 2^{n_\mu} \lambda + X \left(\frac{1}{\lambda} \right)^{\frac{1}{p}}$$

Take the supremum over all cubes Q, then we have,for arbitrary $\lambda > 1$,

$$X \le c_\mu 2^{n_\mu} \lambda + X \left(\frac{1}{\lambda} \right)^{\frac{1}{p}}$$

which implies,if we assume $X < \infty$, then

$$X \le \frac{c_\mu 2^{n_\mu} \lambda}{1 - \left(\frac{1}{\lambda} \right)^{\frac{1}{p}}}$$

Minimizing over $\lambda > 1$, we can get that

$$X \le c_\mu 2^{n_\mu} e(p+1)$$

To remove the hypothesis $X < \infty$, denote the truncation function

$$f_K(x) = \begin{cases} -K, \ f(x) < -K \\ f(x), -K \le f(x) \le K \\ K, K < f(x) \end{cases}$$

And let $X_{\varepsilon,K}=\sup_Q\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{M_{Q,\mu}\left(f_K-(f_K)_{Q,\mu}\right)}{M_\mu^{\#}(f_K)+\varepsilon}\right)d\mu\right)^{\frac{1}{p}}$ for positive constants ε and K.

Note that $|f_K(x)|\le K$, it follows that $M_Q\left(f_K-(f_K)_{Q,\mu}\right)(x)\le 2K$ and so $X_{\varepsilon,K}\le\frac{2K}{\varepsilon}<\infty$.

Fix each pair of ε and K, we can also get that

$$X_{\varepsilon,K}\le c_\mu 2^{n_\mu}e(p+1)$$

For every fixed cube Q, applying the monotone convergence theorem,we obtain that

$$\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{M_{Q,\mu}\left(f-(f)_{Q,\mu}\right)}{M_\mu^{\#}f}\right)^p d\mu\right)^{\frac{1}{p}}=\left(\frac{1}{\mu(Q)}\int_Q\left(\lim_{\substack{\varepsilon\to0\\K\to\infty}}\frac{M_{Q,\mu}\left(f_K-(f_K)_{Q,\mu}\right)}{M_\mu^{\#}(f_K)+\varepsilon}\right)^p d\mu\right)^{\frac{1}{p}}$$

$$=\lim_{\substack{\varepsilon\to0\\K\to\infty}}\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{M_{Q,\mu}\left(f_K-(f_K)_{Q,\mu}\right)}{M_\mu^{\#}(f_K)+\varepsilon}\right)^p d\mu\right)^{\frac{1}{p}}\le c_\mu 2^{n_\mu}e(p+1)$$

Which finishes the proof.

Lemma 3.2 Suppose that (X,υ) is a probability space and f a non-negative function such that for every $1<p<\infty$ we have the L^p bound

$$\left(\int_X f(x)^p d\upsilon(x)\right)^{\frac{1}{p}}\le\gamma p$$

for some constant γ independent from p. Then

$$\upsilon\left(\{x\in X: f(x)>t\}\right)\le ce^{-\frac{t}{4\gamma}},\ t>0.$$

This lemma can be found in[1,Lemma2.3].

Corollary 3.3 There exist some dimensional constants $C_1,C_2>0$ such that

$$\mu\left(\{x\in Q:G_{Q,\mu}(x)>t\}\right)\le C_1e^{-C_2t}\mu(Q),\ t>0.$$

Proof. Let $X=Q$ and $d\upsilon=\frac{d\mu}{\mu(Q)}$, it is easy to verify that (X,υ) is a probability space.

Theorem3.1 yields that the function $G_{Q,\mu}$ satisfies the condition in lemma3.2 with

$$\left(\frac{1}{\mu(Q)}\int_Q G_{Q,\mu}(x)^p d\mu(x)\right)^{\frac{1}{p}}\le c_\mu 2^{n_\mu}e(p+1)\le c_\mu 2^{n_\mu+1}ep$$

Therefore, $\mu\left(\{x\in Q:G_{Q,\mu}(x)>t\}\right)\le C_1e^{-\frac{t}{c_\mu 2^{n_\mu+3}e}}\mu(Q),\ t>0$

where the constant can be taken as $C_2=\frac{1}{c_\mu 2^{n_\mu+3}e}$.

Theorem 3.4 Let μ be a doubling measure in R^n and $Z=\{Z_Q\}$ be a family of (quasi-)Banach spaces with triangle inequality constant uniformly bounded by $K\ge1$. Assume that Z satisfies the $A_\infty(d\mu)$ condition with associated increasing bijection ψ (see Definition 2.5) and is good (see Definition 2.6). Then there exists a constant

$C(\mu,\psi)>0$ such that, for any cube Q in R^n, the following inequality holds

$$\left\|G_{Q,\mu}\chi_Q\right\|_{Z_Q} \le C(\mu,\psi)$$

Where $G_{Q,\mu}(x)=\dfrac{M_{Q,\mu}\left(f-f_{Q,\mu}\right)(x)}{M_\mu^\# f(x)}$ is the same as in theorem 3.1, C_Z is the constant in the $A_\infty(d\mu)$ condition for Z.

Moreover,we can take

$$C(\mu,\psi)=\inf_{\lambda>\max\left\{1,\frac{1}{\psi^{-1}\left(\frac{1}{C_Z\cdot K}\right)}\right\}} \frac{c_\mu 2^{n_\mu}\lambda}{1-C_Z\cdot K\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)}$$

Proof. Let $X=\sup_Q\left\|G_{Q,\mu}\chi_Q\right\|_{Z_Q}$,theorem 3.1 yields that

$$G_{Q,\mu}\chi_Q \le c_\mu 2^{n_\mu}\lambda\chi_Q+\sum_j G_{Q_j,\mu}\chi_{Q_j}$$

for any cube Q in R^n.

By triangle inequality and Definition 2.6,we can obtain that

$$\left\|G_{Q,\mu}\chi_Q\right\|_{Z_Q} \le \left\|c_\mu 2^{n_\mu}\lambda\chi_Q+\sum_j G_{Q_j,\mu}\chi_{Q_j}\right\|_{Z_Q} \le c_\mu 2^{n_\mu}K\lambda+K\left\|\sum_j G_{Q_j,\mu}\chi_{Q_j}\right\|_{Z_Q}$$

$$=c_\mu 2^{n_\mu}K\lambda+K\left\|\sum_j\left\|G_{Q_j,\mu}\chi_{Q_j}\right\|_{Z_{Q_j}}\frac{G_{Q_j,\mu}\chi_{Q_j}}{\left\|G_{Q_j,\mu}\chi_{Q_j}\right\|_{Z_{Q_j}}}\chi_{Q_j}\right\|_{Z_Q} \le c_\mu 2^{n_\mu}K\lambda+KX\left\|\sum_j\frac{G_{Q_j,\mu}\chi_{Q_j}}{\left\|G_{Q_j,\mu}\chi_{Q_j}\right\|_{Z_{Q_j}}}\chi_{Q_j}\right\|_{Z_Q}$$

Since $\left\|\dfrac{G_{Q_j,\mu}\chi_{Q_j}}{\left\|G_{Q_j,\mu}\chi_{Q_j}\right\|_{Z_{Q_j}}}\right\|_{Z_{Q_j}}=1$, according to the $A_\infty(d\mu)$ condition for Z, we have

$\left\|G_{Q,\mu}\chi_Q\right\|_{Z_Q} \le c_\mu 2^{n_\mu}K\lambda+K\cdot X\cdot C_Z\cdot\psi^{-1}\left(\dfrac{\mu\left(\cup_j Q_j\right)}{\mu(Q)}\right)\le c_\mu 2^{n_\mu}K\lambda+K\cdot X\cdot C_Z\cdot\psi^{-1}\left(\dfrac{1}{\lambda}\right)$ Take the supremum over all cubes Q on the left side of the inequality to get that, for any $\lambda>1$,

$$X\le c_\mu 2^{n_\mu}K\lambda+KX\cdot C_Z\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)$$

Therefore, $X\le\dfrac{c_\mu 2^{n_\mu}K\lambda}{1-K\cdot C_Z\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)}$

Since $\lambda>1$ and $1-K\cdot C_Z\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)>0$, we have that

$$\lambda>\max\left\{1,\frac{1}{\psi\left(\frac{1}{C_Z\cdot K}\right)}\right\}$$

Take the infimum over all such λ, we can get the conclusion if $X<\infty$.
Observing that,for each pair for fixed positive constants ε and K, the following inequality holds

$$\left\|\frac{M_{Q,\mu}\left(f_K-(f_K)_{Q,\mu}\right)}{M_\mu^{\#}(f_K)+\varepsilon}\chi_Q\right\|_{Z_Q}\le\frac{2K}{\varepsilon}\|\chi_Q\|_{Z_Q}\le\frac{2K}{\varepsilon}<\infty$$

Similarly,we can obtain that

$$\left\|\frac{M_{Q,\mu}\left(f_K-(f_K)_{Q,\mu}\right)}{M_\mu^{\#}(f_K)+\varepsilon}\chi_Q\right\|_{Z_Q}\le\inf_{\lambda>\max\left\{1,\frac{1}{\psi\left(\frac{1}{C_Z\cdot K}\right)}\right\}}\frac{c_\mu 2^{n_\mu}K\lambda}{1-K\cdot C_Z\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)}$$

By Fatou property from Definition 2.6, we have

$$\|G_{Q,\mu}\chi_Q\|_{Z_Q}=\lim_{\substack{\varepsilon\to0\\K\to\infty}}\left\|\frac{M_{Q,\mu}\left(f_K-(f_K)_{Q,\mu}\right)}{M_\mu^{\#}(f_K)+\varepsilon}\chi_Q\right\|_{Z_Q}\le\inf_{\lambda>\max\left\{1,\frac{1}{\psi\left(\frac{1}{C_Z\cdot K}\right)}\right\}}\frac{c_\mu 2^{n_\mu}K\lambda}{1-K\cdot C_Z\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)}$$

This ends the proof.

Remark 3.5 Theorem 3.1 could be seen as a special case of theorem 3.4.

Note that $\left\{L^p\left(Q,\frac{d\mu}{\mu(Q)}\right)\right\}$ is a family of Banach function spaces and satisfies the condition of Definition 2.6. It also satisfies the $A_\infty(d\mu)$ condition,because:

For any sequence of cubes $\{Q_j\}\in\Delta(Q)$ and family of functions $\{h_j\}$ satisfying $\|h_j\|_{L^p\left(Q_j,\frac{d\mu}{\mu(Q_j)}\right)}=1$, $j\in N$, we have

$$\left\|\sum_j h_j\chi_{Q_j}\right\|_{L^p\left(Q,\frac{d\mu}{\mu(Q)}\right)}=\left(\frac{1}{\mu(Q)}\int_Q\left|\sum_j h_j\chi_{Q_j}\right|^p d\mu\right)^{\frac{1}{p}}=\left(\frac{1}{\mu(Q)}\sum_j\int_{Q_j}|h_j|^p d\mu\right)^{\frac{1}{p}}$$

$$=\left(\sum_j\frac{\mu(Q_j)}{\mu(Q)}\cdot\frac{1}{\mu(Q_j)}\int_{Q_j}|h_j|^p d\mu\right)^{\frac{1}{p}}=\left(\frac{\sum_j\mu(Q_j)}{\mu(Q)}\right)^{\frac{1}{p}},$$

where we can take $C_Z=1,\psi(t)=t^p$.

Remark 3.6 The localized version of the Lorentz quasi-norm could be defined by:

$$\|f\|_{L^{p,q}\left(Q,\frac{d\mu}{\mu(Q)}\right)}=p^{\frac{1}{q}}\left(\int_0^\infty\left[\left(\frac{d_f^Q(s)}{\mu(Q)}\right)^{\frac{1}{p}}s\right]^q\frac{ds}{s}\right)^{\frac{1}{q}}=p^{\frac{1}{q}}\left(\int_0^\infty\left(\frac{d_f^Q(s)}{\mu(Q)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}<\infty$$

where $0<p,q<\infty$, $d_f^Q(s)=\mu\left(\{x\in Q:|f(x)|>s\}\right)$

Note that these localized spaces are quasi-Banach spaces (see[4,Theorem1.4.11]) satisfying the Fatou property (which is easy to check) and $\|f\|_{L^{p,p}\left(Q,\frac{d\mu}{\mu(Q)}\right)}=\|f\|_{L^p\left(Q,\frac{d\mu}{\mu(Q)}\right)}$.

For any cube Q, every $\{Q_j\}\in\Delta(Q)$ and every family of functions $\{h_j\}$ satisfying $\|h_j\|_{Z_{Q_j(d\mu)}}=1$, Since $d^{Q}_{\sum_j h_j \chi_{Q_j}}(s)=\mu\left(\left\{x\in Q:\left|\sum_j h_j(x)\chi_{Q_j}(x)\right|>s\right\}\right)=\sum_j \mu\left(\left\{x\in Q_j:\left|h_j(x)\right|>s\right\}\right)=\sum_j d^{Q_j}_{h_j}(s)$,

it follows that

$$\left\|\sum_j h_j\chi_{Q_j}\right\|_{Z_Q}=p^{\frac{1}{q}}\left(\int_0^\infty\left(\frac{d^{Q}_{\sum_j h_j\chi_{Q_j}}(s)}{\mu(Q)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}=p^{\frac{1}{q}}\left(\int_0^\infty\left(\frac{\sum_j d^{Q_j}_{h_j}(s)}{\mu(Q)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}=p^{\frac{1}{q}}\left(\int_0^\infty\left(\sum_j\frac{\mu(Q_j)}{\mu(Q)}\frac{d^{Q_j}_{h_j}(s)}{\mu(Q_j)}\right)^{\frac{q}{p}}s^{q-1}\right)^{\frac{1}{q}}$$

if $p<q$, $t^{\frac{q}{p}}$ is a convex function,Jensen´s inequality yields that $\left(\sum_j\frac{\mu(Q_j)}{\mu(Q)}\frac{d^{Q_j}_{h_j}(s)}{\mu(Q_j)}\right)^{\frac{q}{p}}\le\sum_j\frac{\mu(Q_j)}{\mu(Q)}\left(\frac{d^{Q_j}_{h_j}(s)}{\mu(Q_j)}\right)^{\frac{q}{p}}$, for each $s\in(0,\infty)$

And then, $p^{\frac{1}{q}}\left(\int_0^\infty\left(\sum_j\frac{\mu(Q_j)}{\mu(Q)}\frac{d^{Q_j}_{h_j}(s)}{\mu(Q_j)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}\le p^{\frac{1}{q}}\left(\int_0^\infty\sum_j\frac{\mu(Q_j)}{\mu(Q)}\left(\frac{d^{Q_j}_{h_j}(s)}{\mu(Q_j)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}$

$$=p^{\frac{1}{q}}\left(\sum_j\frac{\mu(Q_j)}{\mu(Q)}\int_0^\infty\left(\frac{d^{Q_j}_{h_j}(s)}{\mu(Q_j)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}=\left(\frac{\sum_j\mu(Q_j)}{\mu(Q)}\right)^{\frac{1}{q}}.$$

Therefore, the localized Lorentz spaces satisfy the $A_\infty(d\mu)$ condition by taking $C_Z=1,\psi(t)=t^q$.

The average property is also satisfied, since

$\left\|\chi_Q\right\|_{L^{p,q}\left(Q,\frac{d\mu}{\mu(Q)}\right)}=p^{\frac{1}{q}}\left(\int_0^\infty\left(\frac{d^{Q}_{\chi_Q}(s)}{\mu(Q)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}=p^{\frac{1}{q}}\left(\int_0^\infty\left(\frac{\mu(Q)\chi_{(0,1)}(s)}{\mu(Q)}\right)^{\frac{q}{p}}s^{q-1}ds\right)^{\frac{1}{q}}=\left(\frac{p}{q}\right)^{\frac{1}{q}}<1$ Hence, by theorem 3.1, we obtain that

$$\left\|G_{Q,\mu}\right\|_{L^{p,q}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\le\inf_{\lambda>\max\left\{1,\frac{1}{\psi\left(\frac{1}{C_Z\cdot K}\right)}\right\}}\frac{c_\mu 2^{n_\mu}K\lambda}{1-K\cdot C_Z\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)}=\inf_{\lambda>K^q}\frac{c_\mu 2^{n_\mu}K\lambda}{1-K\left(\frac{1}{\lambda}\right)^{\frac{1}{q}}},$$

where we can take $K=2^{\frac{1}{p}}\max\left\{1,2^{\frac{1-q}{q}}\right\}$ (see [4,Proposition1.4.10]).

4. Application in Variable exponent L^p -spaces

In this part, we will present some properties about Variable exponent L^p -spaces with measure μ, the main ideas of these properties come from [5]. Then we can easily obtain that these spaces satisfy the condition of Definition 2.5 and Definition 2.6. Thus we get the application of theorem 3.4 in such spaces.

Lemma 4.1 (Triangle inequality) $\|f+g\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\le\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}+\|g\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$

Proof. Let $\rho_\mu(f)=\frac{1}{\mu(Q)}\int_Q|f(x)|^{p(x)}d\mu(x)$

Since $p(x)\ge1$ for every $x\in R^n$,we can obtain that the function $\phi(t)=t^{p(x)}, t\ge0$ is convex.

By Jensen's inequality,

$\phi(\alpha|f|+\beta|g|)\le\alpha\phi(|f|)+\beta\phi(|g|)$ holds for $\alpha,\beta\ge 0$, $\alpha+\beta=1$.

Therefore,

$$\rho_\mu(\alpha f+\beta g)=\frac{1}{\mu(Q)}\int_Q|\alpha f(x)+\beta g(x)|^{p(x)}d\mu(x)\le\frac{1}{\mu(Q)}\int_Q(\alpha|f(x)|+\beta|g(x)|)^{p(x)}d\mu(x)$$

$$\le\frac{1}{\mu(Q)}\int_Q\left(\alpha|f(x)|^{p(x)}+\beta|g(x)|^{p(x)}\right)d\mu(x)=\alpha\rho_\mu(f)+\beta\rho_\mu(g)$$

Now, fix $\delta_f>\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$ and $\delta_g>\|g\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$, then $\rho_\mu\left(\frac{f}{\delta_f}\right)\le 1$ and $\rho_\mu\left(\frac{g}{\delta_g}\right)\le 1$

Let $\delta=\delta_f+\delta_g$, as

$$\rho_\mu\left(\frac{f+g}{\delta}\right)=\rho_\mu\left(\frac{\delta_f}{\delta}\cdot\frac{f}{\delta_f}+\frac{\delta_g}{\delta}\cdot\frac{g}{\delta_g}\right)\le\frac{\delta_f}{\delta}\rho_\mu\left(\frac{f}{\delta_f}\right)+\frac{\delta_g}{\delta}\rho_\mu\left(\frac{g}{\delta_g}\right)\le 1$$

we can get that $\delta=\delta_f+\delta_g\ge\|f+g\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$

Taking the infimum over all such δ_f and δ_g, we get the conclusion.

Lemma 4.2 For any cube Q in R^n, $\|\chi_Q\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\le 1$, *i.e.* $\left\{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)\right\}$ satisfies the Average property in Definition 2.6.

Proof. By taking $\delta_0=1$, one has

$$\frac{1}{\mu(Q)}\int_Q\left(\frac{|\chi_Q(x)|}{\delta_0}\right)^{p(x)}d\mu(x)=\frac{1}{\mu(Q)}\int_Q(\chi_Q(x))^{p(x)}d\mu(x)=1$$

and thus $1=\delta_0\ge\|\chi_Q\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$.

Lemma 4.3 Let $\{f_k\}$ be a sequence of functions in $L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)$ satisfying $|f_k|\uparrow|f|$, $\mu-a.e.$ then

$\|f_k\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\uparrow\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$, *i.e.* $\left\{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)\right\}$ satisfies Fatou property in Definition 2.6.

Proof. Assume that $f,g\in L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)$ with $|f|\le|g|$, $\mu-a.e.$, then for any δ_g with $\delta_g>\|g\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$,

we have $\rho_\mu\left(\frac{f}{\delta_g}\right)\le\rho_\mu\left(\frac{g}{\delta_g}\right)\le 1$, so, $\delta_g\ge\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$

Take the infimum over all such δ_g, one can get that

$\|g\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\ge\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$

Since $|f_k|\uparrow|f|$, $\mu-a.e.$ therefore, $\|f_k\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\uparrow$ and $\|f_k\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\le\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$

take δ with $\delta<\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$

whether $\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}<\infty$ or $\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}=\infty$, we have $\rho_\mu\left(\frac{f}{\delta}\right)>1$. By the monotone convergence theorem,

$$1<\rho_\mu\left(\frac{f}{\delta}\right)=\frac{1}{\mu(Q)}\int_Q\left(\frac{|f(x)|}{\delta}\right)^{p(x)}d\mu(x)=\frac{1}{\mu(Q)}\int_Q\left(\frac{\lim_{k\to\infty}|f_k(x)|}{\delta}\right)^{p(x)}d\mu(x)=\lim_{k\to\infty}\frac{1}{\mu(Q)}\int_Q\left(\frac{|f_k(x)|}{\delta}\right)^{p(x)}d\mu(x)$$

$$=\lim_{k\to\infty}\rho_\mu\left(\frac{f_k}{\delta}\right)$$

Therefore, $\rho_\mu\left(\frac{f_k}{\delta}\right)>1$ for all k sufficiently large.

Which implies that $\delta\le\|f_k\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$ for all such k.

Hence, $\lim_{k\to\infty}\|f_k\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}=\|f\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}$.

Lemma 4.4 Let Q be a cube in R^n, μ is a nontrivial doubling measure in R^n, then for any $\{Q_j\}\in\Delta(Q)$ and $\{h_j\}$ satisfying $\|h_j\|_{L^p\left(Q_j,\frac{d\mu}{\mu(Q_j)}\right)}=1$, $j\in N$, we have

$$\left\|\sum_j h_j\chi_{Q_j}\right\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\le\left(\frac{\sum_j\mu(Q_j)}{\mu(Q)}\right)^{\frac{1}{p^+}}$$

i.e. $\left\{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)\right\}$ satisfies the $A_\infty(d\mu)$ condition,where $p^+=ess\sup p(x)$.

Proof. Let $\delta_0=\left(\frac{\sum_j\mu(Q_j)}{\mu(Q)}\right)^{\frac{1}{p^+}}$, then we have $\delta_0\ge\left(\frac{\sum_j\mu(Q_j)}{\mu(Q)}\right)^{\frac{1}{p(x)}}$ $\mu-a.e.$

Since $\frac{1}{\mu(Q)}\int_Q\left(\frac{\left|\sum_j h_j(x)\chi_{Q_j}(x)\right|}{\delta_0}\right)^{p(x)}d\mu(x)=\frac{1}{\mu(Q)}\sum_j\int_{Q_j}\left(\frac{|h_j(x)|}{\delta_0}\right)^{p(x)}d\mu(x)=\sum_j\frac{\mu(Q_j)}{\mu(Q)}\int_{Q_j}\left(\frac{|h_j(x)|}{\delta_0}\right)^{p(x)}d\mu(x)$

$$\le\sum_j\frac{\mu(Q_j)}{\mu(Q)}\int_{Q_j}\left(\frac{|h_j(x)|}{\left(\frac{\sum_j\mu(Q_j)}{\mu(Q)}\right)^{\frac{1}{p(x)}}}\right)^{p(x)}d\mu(x)=\sum_j\frac{\mu(Q_j)}{\mu(Q)}\cdot\frac{1}{\frac{\sum_j\mu(Q_j)}{\mu(Q)}}=1$$

We can obtain that $\left\|\sum_j h_j\chi_{Q_j}\right\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)}\le\delta_0=\left(\frac{\sum_j\mu(Q_j)}{\mu(Q)}\right)^{\frac{1}{p^+}}$ with the assumption that $\delta_0>0$. If $\delta_0=0$ or the set

$\{Q_j\}$ is empty,the conclusion is trivial.

Theorem 4.5 Let μ be a nontrivial doubling measure in R^n, then the following inequality holds:

$$\left\|G_{Q,\mu}\chi_Q\right\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)} \le c_\mu 2^{n_\mu} e\left(p^+ +1\right)$$

Where $G_{Q,\mu}(x)$ is defined in theorem3.1, $p^+ = ess\sup p(x)$.

Proof. By theorem 3.4, Lemma 4.1, 4.2, 4.3, 4.4,we can get that

$$\left\|G_{Q,\mu}\chi_Q\right\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)} \le \inf_{\lambda>\max\left\{1,\frac{1}{\psi^{-1}\left(\frac{1}{C_Z\cdot K}\right)}\right\}} \frac{c_\mu 2^{n_\mu}\lambda}{1-C_Z\cdot K\cdot\psi^{-1}\left(\frac{1}{\lambda}\right)}$$

Where we can choose $C_Z=1$ (see Lemma 4.4), $K=1$ (see Lemma4.1), $\psi(t)=t^{p^+}$ (see Lemma 4.4), then we have

$$\left\|G_{Q,\mu}\chi_Q\right\|_{L^{p(\cdot)}\left(Q,\frac{d\mu}{\mu(Q)}\right)} \le \inf_{\lambda>1}\frac{c_\mu 2^{n_\mu}\lambda}{1-\left(\frac{1}{\lambda}\right)^{\frac{1}{p^+}}} = c_\mu 2^{n_\mu} e\left(p^+ +1\right),$$

this concludes the proof of the theorem.

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