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The Influence of *m*-*σ*-embedded Subgroups on the Structure of Finite Groups

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and G is a finite group. A group is said to be *σ-primary* if it is a finite σ_i-group for some i. A subgroup A of G is said to be *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \cdots, t$. A subgroup S of G is *m-σ-permutable* in G if $S = \langle M, B \rangle$ for some modular subgroup M and σ-permutable subgroup B of G. Following this, a subgroup H of G is *m-σ-embedded* in G if there exist an m-σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $HG = HT$ and $H \cap T \leq S \leq H$, where $H^G = \langle H^x | x \in G \rangle$ is the normal closure of H in G. In this paper, we study the structure of G under the condition that some given subgroups of G are m-σ-embedded in G. Some available results are generalized.

Keywords

Finite group, m-σ-embedded subgroup, σ-permutable subgroup, Supersoluble groups, π-nilpotent groups

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. If n is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. $A^G = \bigcap_{x \in G} A^x$ is the core of A in G and $A^G = \langle A^x | x \in G \rangle$ is the normal closure of A in G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. For an integer n, we write $\sigma(n)$ to denote the set $\{\sigma_i | \sigma_i \cap \pi(n) \not= \emptyset\}$; and $\sigma(G) = \sigma(|G|)$.

Following [1, 2], G is said to be *σ-primary* if $|\sigma(G)| \leq 1$; *σ-soluble* if $G = 1$ or $G \neq 1$ every chief factor of G is σ-primary. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some σ_i and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup for every $\sigma_i \in \sigma(G)$. G is said to be *σ-full* if G possesses a complete Hall σ-set; *σ-soluble* if every chief factor of G is σ-primary; *σ-nilpotent* if G has a complete Hall σ-set $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ such that $G = H_1 \times H_2 \times \cdots \times H_t$. Clearly, a σ-nilpotent group is σ-soluble. G is said to be a *σ-full group* of *Sylow type* if every subgroup of G is a D_{σ_i} for all $\sigma_i \in \sigma(G)$. A σ-soluble group is σ-full group of Sylow type (see [3, TheoremsA and B]). A subgroup A of G is said to be *σ-subnormal* [1, 4] in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$.

A subgroup M of G is called *modular* if M is a modular element (in the sense of Kurosh [5,p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of G, that is,

(i) $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$ for all $X \leq G$, $Z \leq G$ such that $X \leq Z$, and

(ii) $\langle M, Y\cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G$, $Z \leq G$ such that $M \leq Z$.

It is well known that embedded subgroups and supplemented subgroups play an importantrole in the theory of finite groups. For example, a subgroup H of G is said to be c-normal [6] in G if G has a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H. A subgroup H of G is called *n*-embedded [7] in G if G has a normal subgroup T such that $H^G = HT$ and $H\cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generatedby all those subgroups of H which are *s*-permutable in G (note that a subgroup A of G is saidto be *s*-permutable in G if $AP = PA$ for any Sylow subgroup P of G). A subgroup $H$of G is called σ-permutable [1] in$G$ if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^xH$ forall $A \in \mathcal{H}$ and all $x \in G$. Note that in the case when $\sigma = \{\{2\},\{3\},\cdots\}$, then a σ-permutable subgroup is just an *s*-permutable subgroup. Wu et al. in [8] introduced the definition of *n*-σ-embedded subgroups of finite groups: a subgroup $H$of G is said to be *n-σ-embedded* in G if there exists a normal subgroup T of G such that $H^G = HT$ and $H\cap T \leq H_{\sigma G}$ .Wei [9] proposed the concept of *m-σ-permutable* and weakly *m*-σ-permutable subgroups offinite groups. A subgroup H of G is said to be: *m-σ-permutable* in Gif $H = \langle A, B\rangle$ forsome modular subgroup $A$and σ-permutable subgroup B of G; weakly *m-σ-permutable* in Gif there are an *m*-σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $G = HT$ and $H\cap T \leq S \leq H$.

In 2021, Guo et al. in [10] gave the following definition:

Definition 1.1. A subgroup H of a group G is said to be *m*-σ-embedded in G if there are an *m*-σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $H^G = HT$ and $H\cap T \leq S \leq H$.

Some properties of *m*-σ-embedded subgroups were analyzed in [10]. In this paper, we continue the research of *m*-σ-embedded subgroups. Firstly, we obtain the following result:

Theorem 1.2. *Let G be a σ-full group of Sylow type and H is a Hall σ_i-subgroup of G. Supposethat the smallest prime p of π(G) belongs to σ_i. If H is m-σ-embedded in G, then G is σ-soluble.*

The following results immediately appear from Theorem 1.2.

Corollary 1.3. (See Zhang et al. [11, Thoerem 1.4])

Let G be a σ-full group of Sylow type, and suppose that every Hall σ_i-subgroup of G is weakly σ-permutable in G for every $\sigma_i \in \sigma(G)$. Then G is σ-soluble.

Corollary 1.4. (See Cao et al. [12, Thoerem 1.1])

Let G be a σ-full group of Sylow type, let $H \neq 1$ be a Hall σ_i-subgroup of G, and let thesmallest prime p of $\pi(G)$ belongs to σ_i. If H is weakly σ-permutable in G, then G is σ-soluble.

Corollary 1.5. (See Guo et al. [10, Thoerem 1.1])

Let G be a σ-full group of Sylow type, and suppose that every Hall σ_i-subgroup of G is*m*-σ-embedded in G for every $\sigma_i \in \sigma(G)$. Then G is σ-soluble.

Recall a class of groups is a *formation* if for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$. For a formation $\mathfrak{F}$, a chief factor H/K of G is $\mathfrak{F}$-*central* in G if$(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. We use $Z_{\mathfrak{F}}(G)$ to denote the product of all normal subgroups A of G such that either $A = 1$ or $A \neq 1$ and every chief factor of G below A is $\mathfrak{F}$-central in G. $Z_{\mathfrak{F}}(G)$ is called that the $\mathfrak{F}$-*hypercentre* of G. We use $\mathfrak{U}$ to denote the class of all supersolublegroups.

Following [13, 14], we use $\mathcal{L}_{c\mathfrak{F}}(G)$ to denote the set of all subgroups A of G such that everychief factor H/K of G between A_G and A^G is $\mathfrak{F}$-central in G. Recall that a subgroup $H$of G is said to be a $c\mathfrak{F}$-normal subgroup of G if $H \in \mathcal{L}_{c\mathfrak{F}}(G)$, that is, every chief factor L/K of Gbetween H_G and H^G is $\mathfrak{F}$-central in G(see [10, Definition 1.10]).

Combining with $c\mathfrak{F}$-normal subgroups of G, we obtain some new characterizations of supersoluble groups and σ_j-nilpotent groups (that is, G is *p*-nilpotent for any prime $p \in \sigma_j$).

Theorem 1.6. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ a completeHall σ-set of Gsuch that H_i is a supersoluble σ_i-subgroup for all $i = 1,\ldots,t$. If everymaximal subgroup of any non-cyclic H_i is both m-σ-embedded and c𝔘-normal in G, then G issupersoluble.*

The following results directly follow from Theorem 1.6.

Corollary 1.7. (See Guo et al. [10, Thoerem 1.11])

Let G be a σ-full group of Sylow type, and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a complete Hall σ-set of$G$ such that H_i is a nilpotent σ_i-subgroup for all $i = 1,\ldots,t$. If every maximal subgroup of any non-cyclic H_i is both *m*-σ-embedded and $c\mathfrak{U}$-normal in G, then G is supersoluble.

Theorem 1.8. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ a complete Hallσ-set of G such that H_i is a supersoluble σ_i-subgroup of G for all i = 1, . . . , t. Suppose that for some $\sigma_i \in \sigma(G)$, $(|G|, p-1) = 1$ for any prime $p \in \sigma_j$. If every maximal subgroup of $H_j$is both m-σ-embedded and c𝔘-normal in G, then G is σ_j-nilpotent.*

2. Preliminaries

Let Π be a subset of the set σ. A natural number n is said to be a Π-number if $\sigma(n) \subseteq \Pi$.

A subgroup A of G is said to be: a *Hall Π-subgroup* of G[1] if $|A|$ is a Π-number and $|G : A|$is a Π' -number; a *σ-Hall subgroup* of G if A is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$. $O_{\Pi}(G)$to denote the subgroup of G generated by all its normal Π-subgroups. Instead of $O_{\{\sigma_i\}}(G)$ wewrite $O_{\sigma_i}(G)$.

Lemma 2.1. (See [1, Lemma 2.6]) Let A, $K$and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G. Then:

(1) $A \cap K$ is σ-subnormal in K.

(2) If K is a σ-subnormal subgroup of A, then K is σ-subnormal in G.

(3) AN/N is σ-subnormal in G/N.

(4) If $N \leq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.

(5) If A is a σ-Hall subgroup of G, then A is normal in G.

Lemma 2.2. (See [1, Lemma 2.1]) The class of all σ-soluble groups is closed under takingdirect products, homomorphic images and subgroups. Moreover, any extension of a σ-solublegroup by a σ-soluble group is also a σ-soluble group.

Lemma 2.3. (See [9, Lemma 2.5]) Let A, B and N be subgroups of G, where A is *m*-σ-permutable in G and $N$is normal in G. Then:

(1) AN/N is *m*-σ-permutable in G/N.

(2) If G is a σ-full group of Sylow type and $A \leq B$, then A is *m*-σ-permutable in *B*.

(3) If G is a σ-full group of Sylow type, $N \leq B$ and B/N is *m*-σ-permutable in G/N, then B is *m*-σ-permutable in G.

Lemma 2.4. (See [10, Lemma 2.4]) Let G be a σ-full group of Sylow type and $H \leq K \leq G$.

(1) If H is *m*-σ-embedded in G, then H is *m*-σ-embedded in K.

(2) Suppose that H is normal in G. Then K/H is *m*-σ-embedded in G/H if and only if K is *m*-σ-embedded in G.

(3) Suppose that H is normal in G. Then for every *m*-σ-embedded subgroup E of G with $(|H|, |E|) = 1$, HE/H is *m*-σ-embedded in G/H.

We call the product of all normal σ-soluble subgroups of G the σ-radical of G and denoteit by $R_\sigma(G)$ and $R_\sigma(G)$ is σ-soluble (see [3]).

Lemma 2.5. (See [3, Lemma 2.3]) If H is a subnormal σ-soluble subgroup of G, then $H \leq R_\sigma(G)$.

Lemma 2.6. (See [5, Theorem 5.2.5]) If H is a modular subgroup of G, then every chief factor of G between H^G and H_G is cyclic.

From Lemma 3.1 in [13], we have the following lemma:

Lemma 2.7. Let A be a $c\mathfrak{F}$-normal subgroup of G and Na normal subgroups of G. Then AN/N is a $c\mathfrak{F}$-normal subgroup of G/N. In particular, when $\mathfrak{F} = \mathfrak{U}$, we have that if A is a $c\mathfrak{U}$-normal subgroup of G, then AN/N is a $c\mathfrak{U}$-normal subgroup of G/N.

3. Proof of the results

Proof of Theorem 1.2. Assume that this is false and let (G, H) be a counterexamplewith minimal $|G| + |H|$. Then $|\sigma(G)| > 1$, and $p = 2 \in \sigma_i$ by the well-known Feit-Thompsontheorem. Without loss of generality, we may assume that $i = 1$, that is, H is a Hall σ_1-subgroup of G. By the hypothesis, there exist an *m*-σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $HT = (H)^G$ and $H \cap T \leq S \leq H$. Then $S = \langle A, B \rangle$ for some modular subgroup A and σ-permutable subgroup B of G.

(1) *G/N is σ-soluble for every non-identity normal subgroup N of G.*

By Feit-Thompson theorem, we may assume that $2||G/N|$. Then $L/N = HN/N$ is a Hall σ_1-subgroup of G/N. Since $|HN \cap T : H \cap T| = |(HN \cap T)H : H|$ is a $\sigma_1{}'$ -number, $(|HN \cap T : H \cap T|, |HN \cap T : N \cap T|) = 1$. Hence $HN \cap T = (H \cap T)(N \cap T)$ by [15, A,Lemma 1.6]. Consequently, $(HN/N) \cap (TN/N) = (HN \cap T)N/N = (H \cap T)N/N \leq SN/N \leq HN/N = L/N$ where SN/N is *m*-σ-permutable in G/N by Lemma 2.3(1). By Lemma 2.1(3), TN/N is σ-subnormal in G/N. It is also clear that $(L/N)^{G/N} = (HN/N)^{G/N} = (H)^G N/N = (HN/N)(TN/N) = (L/N)(TN/N)$. Therefore L/N is *m*-σ-embedded in G/N. This showsthat $(G/N, L/N)$ satisfies the hypothesis. The choice of (G, H) implies that G/N is σ-soluble.

(2)$T \neq 1$.

If $T = 1$, then $H = (H)^G$ is normal in G. Hence G/H is σ-soluble by Claim (1). By Lemma 2.2, G is σ-soluble, a contradiction. Hence we have Claim (2).

(3) *G is not a simple group.*

Suppose that G is an non-abelian simple group. Then 1 and G are the only two σ-subnormal subgroup of G. Claim (2) shows that $T = G$, and so $H = S$. Assume that $A \neq 1$,then, clearly, $A_G = 1$. It follows from Lemma 2.6 that A^G is hypercyclic embedded in G.This contradiction shows that $A = 1$. Then $H = S = B$ is σ-permutable in G. So H is σ-subnormal in G by [1, Theorem B]. Therefore H is normal in G by Lemma 2.1(5). It is follows from $|\sigma(G)| > 1$ that $H = 1$, a contradiction. Hence we have (3).

(4) *Let R be a minimal normal subgroup of G, then R is the unique minimal normalsubgroup of G. Moreover R is not σ-soluble.*

By Claim (2), we have that there exists a minimal normal subgroup R of G. Assume that$G$ has another minimal normal subgroup N of G such that $R \neq N$. By Claim (1), we know that G/R and G/N are both σ-soluble. Then $G \simeq G/(R \cap N)$ is σ-soluble by Lemma 2.2, acontradiction. Hence we have (4).

(5)$S \neq 1$.

Assume that $S = 1$. Then $H \cap T = 1$, and so T is a σ-subnormal Hall $\sigma_i{}'$ -subgroup of H^G. Hence T is a normal complement to H in H^G by Lemma 2.1(5). Moreover, $H^G/T \simeq H$ is σ-soluble. Since $2 \nmid |T|$, T is soluble by Feit-Thompson theorem. Therefore H^G is σ-solubleby Lemma 2.2. By Claim (4), we have that $R \leq H^G$ and so R is σ-soluble, a contradiction to Claim (4). Hence we have (5).

(6) *The final contradiction.*

We claim that $H_G \neq 1$. If $H_G = 1$, then every non-identity subgroup K of H is not σ-permutable in G since otherwise for every $x \in G$ we have that $KH^x = H^xK = H^x$, which implies that $1 < K \leq H_G = 1$. Hence $B = 1$ and so $S = A$ is a modular subgroup of G with $S_G = 1$. But since $S \neq 1$ by Claim (5), we have that $1 < S^G$ is hypercyclically embedded in G by Lemma 2.6 and so R is cyclic by Claim (4), contrary to Claim (4). Therefore $H_G \neq 1$. In view of Claim (4) again, we have that $R \leq H_G \leq H$ is σ-soluble, a contradiction. This contradiction completes the proof of the Theorem.

Proof of Theorem 1.6. Assume that this theorem is false and let G be a counterexample of minimal order. Then $|\sigma(G)| > 1$.

Let p be the smallest prime dividing $|G|$. Without loss of generality, we may assume that $p \in \sigma_1$. Let R be a minimal normal subgroup of G.

(1) *If R is σ-primary, then G is soluble and G/R is supersoluble. Hence R is not cyclic.*

We show that the hypothesis holds for G/R. First note that $\{H_1R/R, \dots, H_tR/R\}$ is a complete Hall σ-set of G/R, where $H_iR/R \simeq H_i/H_i \cap R$ is supersoluble since H_i is supersoluble by hypothesis.

Now let V/R be a maximal subgroup of H_iR/R, so $|(H_iR/R) : (V/R)| = q$ for some prime q. It is clear that $V = V \cap H_iR = (V \cap H_i)R$. Hence

$$q = |(H_iR/R) : (V/R)| = |(H_iR/R) : ((V \cap H_i)R/R)|$$
$$= |H_iR : (V \cap H_i)R| = |H_i||R||(V \cap H_i) \cap R|/|V \cap H_i||R||H_i \cap R|$$
$$= |H_i|/|V \cap H_i| = |H_i : (V \cap H_i)|$$

so$V \cap H_i$ is a maximal subgroup of H_i. Assume that $H_iR/R$is not cyclic. Then H_i is not cyclic, so $V \cap H_i$ is m-σ-embedded and $c\mathfrak{U}$-normal in G by hypothesis. Moreover, since R is a σ_k-group for some k and hence $R \leq V \cap H_i$ in the case when $i = k$ and$(|R|, |V \cap H_i|) = 1$ in the case when $i = k$since $R \leq H_k$. Then $V/R = (V \cap H_i)R/R$ is m-σ-embedded in G/R by Lemma 2.4(2)(3). And by Lemma 2.7, we have $V/R = (V \cap H_i)R/R$ is $c\mathfrak{U}$-normal in G/R. Hence the hypothesis holds for G/R, so the choice of G implies that G/R is supersoluble and G is σ-soluble. Finally, since $R \leq H_k$, R is soluble and so G is soluble. Hence we have (1).

(2) *G is σ-soluble. Hence G is soluble.*

Suppose that this is false. Then $(H_1)_G = 1$ and $O_{\sigma k}(G) = 1$ for all $\sigma_k \in \sigma(G)$ by Claim (1). Moreover, a Sylow p-subgroup $P$of H_1 is not cyclic. Indeed, if P is cyclic, then G is p-nilpotent by [16, IV, 2.8] and so G is soluble by the Feit-Thompson theorem since p is the smallest prime dividing $|G|$.

Since $H_1$is supersoluble and P is not cyclic by above, we obtain that $H_1 = V_1V_2$ for some maximal subgroups V_1, V_2 of H_1 with $|H_1 : V_1| = p$ and V_i is m-σ-embedded in G for $i = 1, 2$. Hence there exists σ-subnormal subgroups T_i and some m-σ-permutable subgroups S_i of G such that $V_iT_i = (V_i)^G$ and $V_i \cap T_i \leq S_i \leq V_i$ for $i = 1, 2$. Then $(S_i)_G \leq (H_1)_G = 1$ and $S_i = \langle A_i, B_i \rangle$ for some modular subgroup A_i and σ-permutable subgroup B_i of G with $(A_i)_G = 1 = (B_i)_G$. Since G has no cyclic minimal normal subgroup by Claim (1), Lemma 2.6 implies that

$A_i = 1$, so $S_i = B_i$ is σ-permutable in G. If $S_i > 1$, then we have $1 < S_i \leq O_{\sigma_1}(G)$ by [1, Theorem B(i)] and Lemma 2.2(8) in [11], a contradiction. Therefore $S_i = 1$, which implies that $T_i \cap V_i = 1$ and so $|(T_i)_{\sigma_1}|$ divides p, that is $|(T_i)_{\sigma_1}| = 1$ or $|(T_i)_{\sigma_1}| = p$. If $|(T_i)_{\sigma_1}| = 1$, then T_i is a Hall $\sigma_1{}'$ -subgroup of $(V_i)^G$. Hence T_i is normal in $(V_i)^G$ and so normal in G. Moreover, since $(V_i)^G/T_i$ is a σ_1-group and T_i is σ-soluble, $(V_i)^G$ is σ-soluble. Hence $V_i \leq (V_i)^G \leq R_\sigma(G)$ by Lemma 2.5. If $|(T_i)_{\sigma_1}| = p$, then T_i is p-nilpotent. Hence T_i has a normal Hall p'-subgroup K_i of T_i. Assume that $K_i = 1$. Then $(V_i)^G$ is a σ_1-subgroup, and so $V_i \leq (V_i)^G \leq R_\sigma(G)$ by Lemma 2.5. Assume that $K_i \neq 1$. Since T_i is σ-subnormal in G, K_i is σ-subnormal in G and so K_i is σ-subnormal in $(V_i)^G$. It follows that Lemma 2.1(5) that $K_i$is normal in $(V_i)^G$. Besides, since $(V_i)^G/K_i$ is a σ_1-group and K_i is soluble, $(V_i)^G$ is σ-soluble. Hence $V_i \leq (V_i)^G \leq R_\sigma(G)$ by Lemma 2.5. By above, we have that $H_1 = V_1V_2 \leq \langle (V_1)^G, (V_2)^G \rangle \leq R_\sigma(G)$, and so$|G/R_\sigma(G)|$ is a $\sigma_1{}'$-number, which means that G is σ-soluble. The contradiction shows that (2) holds.

(3) *R is the unique minimal normal subgroup of G, $R = O_q(G) \nleq \Phi(G)$ for some prime q and $|R| > q$.*

By Claim (2), G is soluble and so R is a q-group for some prime q. Hence the choice of G and Claim (1) imply that R is the unique minimal normal subgroup of G and $|R| > q$. Moreover, $R \nleq \Phi(G)$ by [16, VI, 8.6], so $R = C_G(R) = O_q(G)$ by [15, A, 15.2].

The final contradiction. Let $q \in \sigma_i$. Then $R \leq H_i$ and for some maximal subgroup M of G, we have that $G = R \rtimes M$ by Claim (3). Hence $H_i = R \rtimes (H_i \cap M)$. Since H_i is supersoluble by hypothesis, some maximal subgroup $W$of R is normal in H_i. Then $V = W(H_i \cap M)$ is a maximal subgroup of H_i. Hence V is $c\mathfrak{U}$-normal in G by hypothesis. Claim (3) implies that $V_G = 1$. Hence $V^G \leq Z_{\mathfrak{U}}(G)$ and so $R \leq Z_{\mathfrak{U}}(G)$ by Claim (3). It follows from Claim (1) that G is supersoluble. This contradiction completes the proof of the result.

Proof of Theorem 1.8. Without loss of generality, we may assume that $j = 1$. Assume that this is false and let (G, H_1) be a counterexample with $|G| + |H_1|$ minimal. Then:

(1) *G is soluble.*

This directly follows by a similar argument as Claims (1)(2) in the proof of Theorem 1.6.

(2) $O_{\sigma_1{}'}(G) = 1$.

Assume that this is false. Let $N = O_{\sigma_1{}'}(G)$. Clearly, $\sigma_1 \in \sigma(G/N)$ and H_1N/N is a Hall σ_1-subgroup of G/N. Since $(|G|, p-1) = 1$ for any $p \in \sigma_1$, we have that $(|G/N|, p-1) = 1$. Let V/N be any maximal subgroup of H_1N/N. Then $V = (V \cap H_1)N$ is a maximal subgroup of H_1N. With a same discussion as Claim (1) in the proof of Theorem 1.6, we have that $V \cap H_1$ is a maximal subgroup of H_1. Then by the hypothesis and Lemma 2.4(3), V/N is *m-σ*-embedded in G/N. And by Lemma 2.7, we have V/N is $c\mathfrak{U}$-normal in G/N. This shows that $(G/N, H_1N/N)$ satisfies the hypothesis, and so G/N is σ_1-nilpotent by the choice of (G, H_1). It follows that G is σ_1-nilpotent, a contradiction. Hence we have (2).

(3) *Let R be a minimal normal subgroup of G. Then R is an elementary abelian p-group for some prime $p \in \sigma_1$ and G/R is σ_1-nilpotent.*

Claims (1) and (2) imply that R is an elementary abelian p-group for some prime $p \in \sigma_1$, and so $R \leq H_1$. If $R = H_1$, then G/R is a $\sigma_1{}'$-group and hence G/R is σ_1-nilpotent. Now, we consider that $R \neq H_1$. Then $\sigma_1 \in \sigma(G/R)$ and H_1/R is a Hall σ_1-subgroup of G/R. Since $(|G|, p-1) = 1$ for any $p \in \sigma_1$, we have that $(|G/R|, p-1) = 1$.

Let W/R be a maximal subgroup of H_1/R. Then W is a maximal subgroup of H_1. Hence by the hypothesis and Lemma 2.4(2), W/R is *m-σ*-embedded inG/R. And by Lemma 2.7, we have W/R is $c\mathfrak{U}$-normal in G/R. This shows that $(G/R, H_1/R)$ satisfies the hypothesis. Therefore G/R is σ_1-nilpotent by the choice of (G, H_1).

(4) *R is the unique minimal normal subgroup of G, $\Phi(G) = 1$ and $R = O_p(G) = F(G) = C_G(R)$.*

This directly follows from Claims (1), (3) and [15, Chapter A, Theorem 15.2].

(5) *R is non-cyclic.*

Assume that R is cyclic. Then R is a cyclic group of order p by Claim (3). It follows that $G/R = N_G(R)/C_G(R) \lesssim Aut(R)$ is a cyclic group of order $p - 1$. But as $(|G|, p-1) = 1$, $G = R$ is a cyclic group of order p, a contradiction. Hence we have (5).

(6) *Final contradiction*.

Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = R \rtimes M$. It follows that $H_1 = R \rtimes (H_1 \cap M)$. Since H_1 is supersoluble by hypothesis, some maximal subgroup $W$of R is normal in H_1. Then $V = W(H_1 \cap M)$ is a maximal subgroup of H_1. Hence $V$is $c\mathfrak{U}$-normal in G. Claim (4) implies that $V_G = 1$. Hence $V^G \leq Z_{\mathfrak{U}}(G)$ and so $R \leq Z_{\mathfrak{U}}(G)$ by Claim (4), contrary to Claim (5). This completes the proof.

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