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Methods and Skills of Solving Several Kinds of Differential Mean Value Theorem Proving Problems

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Abstract

The differential mean value theorem is one of the important basic theorems in differential calculus and a powerful tool to study the properties of the Han script. It reflects the relationship between functions and derivatives. Many proof questions encountered in the study of calculus need to use the relevant knowledge of the differential mean value theorem, and students are often at a loss. In this paper, several types of differential mean value theorems are discussed in depth, and it is found that the key point and difficulty in proving many propositions related to differential mean value theorems is to construct appropriate auxiliary functions. By constructing auxiliary functions, abstract and seemingly unrelated conclusions can be transformed into relatively straightforward views, which can easily find the path that meets the conditions of differential mean value theorems, and then solve the problem. Through these types of questions, this paper summarizes the common skills to find auxiliary functions, hoping to provide some help for students in learning, postgraduate entrance examination, etc.

Keywords

Rolle's theorem, Lagrange mean value theorem, Cauchy mean value theorem, Taylor formula

1. Differential mean value theorem

The differential mean value theorem mainly includes Rolle'stheorem, Lagrange's mean value theorem and Cauchy's mean value theorem. Among them, Lagrange's mean value theorem gives a certain relationship between derivatives and functions, which is often used to prove some equations and inequalities. Rolle's theorem is a special case of Lagrange's mean value theorem [1]. It is often used to prove equations or solve the existence of some equation roots in combination with the zero point theorem. Cauchy mean value theorem is the extension of Lagrange mean value theorem to multiple functions, which can be used to prove equality or inequality.

Rolle Theorem: If $f(x)$ satisfy: (1) Continuous on $[a,b]$; (2) Differentiable on (a,b); (3) $f(a)=f(b)$. Then there is at least one point ξ on (a,b), such that $f'(\xi)=0$.

Lagrange Mean Value Theorem: If $f(x)$ satisfy: (1) Continuous on $[a,b]$; (2) Differentiable on (a,b). Then there is at least one point ξ on (a,b), such that $f(b)-f(a)=f'(\xi)(b-a)$.

Cauchy Mean Value Theorem: If $f(x)$, $g(x)$ satisfy: (1) Continuous on $[a,b]$; (2) Differentiable on (a,b); (3) For any $x \in (a,b)$, $F(x) \neq 0$. Then there is at least one point ξ on (a,b), such that $\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(\xi)}{F'(\xi)}$.

2. Related questions

Type 1: Proof of continuous function proposition on closed interval.

The common method for this type of questions is to construct appropriate auxiliary functions according to different purposes to achieve the goal of proof. Main steps: (1) Construct auxiliary function $F(x)$; (2) If there is no integral operation in the process of doing $F(x)$, then verify that $F(x)$ satisfy the zero point theorem is satisfied (3) If there is integral operation in the process of doing $F(x)$, then verify that $F(x)$ satisfy the Rolle's theorem [2]

Commonly used as auxiliary function:

(1) Rewrite ξ or x_0 in the conclusion into x, transposition, Make one end of the equation is 0, The other end is recorded $F^*(x)$.

(2) Order $F(x) = F^*(x)$, verify whether $F(x)$ satisfy the zero point theorem . If satisfied, the proposition is proved; If not, change the order $F'(x) = F^*(x) \Rightarrow F(x) = \int F^*(x)\mathrm{d}x + C$ (Order $C = 0$), or $F'(x) = k(x)F^*(x) \Rightarrow F(x) = \int k(x)F^*(x)\mathrm{d}x$ (Where $k(x)$ is a known function).

(3) Verify whether $F(x)$ satisfies Rolle's theorem. If satisfied, the proposition is proved; If not, change the order $F''(x) = F^*(x)$ (after two times of integration, the result can be obtained).

(4) Expand $F(x)$ into the first-order Taylor formula at the specified point, and the proposition can be proved.

Example 1 Let the function $f(x)$ be continuous on $[a,b](a<b)$, and $f(x) > 0$. Prove that there is at least one point $\xi \in (a,b)$, can be used $\int_a^\xi f(x)dx = \int_\xi^b f(x)dx = \frac{1}{2}\int_a^b f(x)dx.$ [1]

Analyse: $\int_a^\xi f(x)dx - \int_\xi^b f(x)dx = 0 \Rightarrow F(x) = \int_a^x f(x)dx - \int_x^b f(x)dx$

Proof: Introducing auxiliary functions $F(x) = \int_a^x f(x)dx - \int_x^b f(x)dx$, then $F(x) \in C[a,b]$, $F(a) = -\int_a^b f(x)dx < 0$, $F(b) = \int_a^b f(x)dx > 0$ (because of $f(x) > 0$).

That is $F(a)\cdot F(b) < 0$, according to the zero point theorem, there is at least one point $\xi \in (a,b)$ can be used $F'(\xi) = 0$. That is $\int_a^\xi f(x)dx = \int_\xi^b f(x)dx = \frac{1}{2}\int_a^b f(x)dx.$

Example 2 Let $f(x)$ is differentiable on $[0,1]$, and $f(1) - 2\int_0^{\frac{1}{2}} xf(x)dx = 0$, then there is $\xi \in (0,1)$ such that $f'(\xi) = -\frac{f(\xi)}{\xi}$.

Analyse: The conclusion is available $\xi f'(\xi) + f(\xi) = 0 \Rightarrow F(x) = xf(x)$, obvious $F(0) = 0$. Equivalent to proving existence $\xi \in (0,1)$ bring $F'(\xi) = 0$. Therefore, just find two points in $(0,1)$ so that the function values of $F(x)$ at these two points are equal, then using Rolle's theorem.

Proof: Introducing auxiliary functions $F(x) = xf(x)$, $2\int_0^{\frac{1}{2}} xf(x)dx$ is the integral mean of $F(x) = xf(x)$ over $\left[0,\frac{1}{2}\right]$, from the first mean value theorem of integral, $\exists x_0 \in \left(0,\frac{1}{2}\right)$, bring $F(x_0) = 2\int_0^{\frac{1}{2}} xf(x)dx = f(1) = F(1)$.

Then $F(x)$ satisfies Rolle's theorem on $[x_0,1]$, $\exists \xi \in (x_0,1) \subset (0,1)$, bring

$F'(\xi)=\xi f'(\xi)+f(\xi)=0$. That is $f'(\xi)=-\dfrac{f(\xi)}{\xi}$.

Example 3 Let $f(x),g(x)\in C[a,b]$, And is internally derivable on (a,b), $f(a)=f(b)=0$. Prove that there is $\xi\in(a,b)$ such that $f'(\xi)+f(\xi)g'(\xi)=0$.

Analyse: $f'(x)+f(x)g'(x)=0\Rightarrow\dfrac{f'(x)}{f(x)}+g'(x)=0$.

$\Rightarrow\left[\ln f(x)\right]'+\left[\ln e^{g(x)}\right]'=0\Rightarrow\left[\ln f(x)e^{g(x)}\right]'=0$, order $F(x)=f(x)e^{g(x)}$.

Proof: Introducing auxiliary functions $F(x)=f(x)e^{g(x)}$, $F(x)$ satisfy the conditions of Rolle's theorem, and $F(a)=F(b)=0$. So we know from Rolle's theorem, $\exists\xi\in(a,b)$, bring $F'(\xi)=0$, that is $f'(\xi)+f(\xi)g'(\xi)=0$.

Example 4 Let $f(x)\in C[0,1]$, and $f(0)=\int_0^1 f(x)dx=0$. Prove that there is $\xi\in(0,1)$, bring $\xi f(\xi)=\int_0^{\xi} f(x)dx.$

Analyse: Order $\varphi(x)=xf(x)-\int_0^x f(t)dt$, although $\varphi(x)\in C[0,1]$, but $\varphi(0)\varphi(1)<0$ does not hold, zero point theorem cannot be used. $\varphi(x)=xf(x)-\int_0^x f(t)dt\Rightarrow\dfrac{xf(x)-\int_0^x f(t)dt}{x^2}=\left[\dfrac{\int_0^x f(t)dt}{x}\right]'$.

Therefore, it is used as an auxiliary function $F(x)=\begin{cases}\dfrac{\int_0^x f(t)dt}{x}, x\neq 0\\ 0, x=0\end{cases}$

Proof: Introducing auxiliary functions $F(x)=\begin{cases}\dfrac{\int_0^x f(t)dt}{x}, x\neq 0\\ 0, x=0\end{cases}$,

$$\lim_{x\to 0}F(x)=\lim_{x\to 0}\frac{\int_0^x f(t)dt}{x}=\lim_{x\to 0}\frac{f(x)}{1}=f(0)=0=F(0)$$

So that $F(x)\in C[0,1]\bigcap D(0,1)$, and $F(1)=\dfrac{\int_0^1 f(t)dt}{1}=0=F(0)$, according to Rolle's theorem, at least exists $\xi\in(0,1)$, so that $F'(\xi)=0$, that is $\xi f(\xi)=\int_0^{\xi} f(x)dx.$

Type 2: Proof of proposition $f^{(n)}(\xi)=0$.

This type of question needs to start from the conclusion and verify that ξ is the maximum or extreme point of $f^{(n-1)}(x)$. It can be proved by using the necessary conditions for the existence of extremum or fermat lemma, or verify that $f^{(n-1)}(x)$ satisfies the Rolle's theorem condition on the interval included in $x=\xi$ [2]

Example 5 Let $f(x)\in C[0,3]\bigcap D(0,3)$, and $f(0)+f(1)+f(3)=3, f(3)=1$. Prove that there is $\xi\in(0,3)$, bring $f'(\xi)=0$.

Analyse: According to the conclusion of the proposition, it is necessary to find a suitable interval to satisfy the conditions of Rolle's theorem.

Proof: Set knowledge by questions, $f(x)\in C[0,2]$, so that $m\le f(x)\le M$, $m\le f(0)\le M$, $m\le f(1)\le M$, $m\le f(2)\le M$, then $3m\le f(0)+f(1)+f(2)\le 3M\Rightarrow m\le\dfrac{f(0)+f(1)+f(2)}{3}\le M$.

According to the intermediate value theorem, $\exists\eta\in(0,2)$, so that $f(\eta)=\frac{f(0)+f(1)+f(2)}{3}=1$.

Also $f(3)=1$, it can be seen that Rolle's theorem is satisfied on $[\eta,3]$, there is a point $\xi\in(\eta,3)$, that makes $f'(\xi)=0$.

Example 6 Let $f(x)\in C[0,2]$, second order differentiability in $(0,2)$, $f(0)=f(1)$, $f(2)=2\int_1^{\frac{3}{2}}f(x)dx$. Prove that there is $\xi\in(0,2)$, bring $f''(\xi)=0$.

Analyse: According to the conclusion of the proposition, we need to use Rolle's theorem to prove, then we need to find the interval that meets the conditions of Rolle's theorem.

Proof: $\because f(0)=f(1)$, it is known that $f(x)$ satisfies Rolle's theorem on $[0,1]$, then $\exists\xi_1\in(0,1)$, bring $f'(\xi_1)=0$.

It is also known from the integral mean value theorem, $\exists\eta\in\left(1,\frac{3}{2}\right)$, bring $f(2)=2\int_1^{\frac{3}{2}}f(x)dx=2\left(\frac{3}{2}-1\right)f(\eta)$.

From the above formula, $f(x)$ satisfies Rolle's theorem on $[\eta,2]$, then $\exists\xi_2\in(\eta,2)$, bring $f'(\xi_2)=0$.

From $f'(\xi_1)=f'(\xi_2)=0$, $f'(x)$ is derivable on $(0,2)$, It is known that $f'(x)$ satisfies Rolle's theorem on $[\xi_1,\xi_2]$, so that $\exists\xi\in(\xi_1,\xi_2)\subset(0,2)$, bring $f''(\xi)=0$.

Type 3: To prove the conclusion "At least a point $\xi\in(a,b)$ exists, such that $f^{(n)}(\xi)=k(k\neq 0)$, orThe algebraic formula formed by $a,b,f(a),f(b),\xi,f(\xi),f'(\xi),\cdots,f^{(n)}(\xi)$ holds."

It is often necessary to introduce auxiliary functions $F(x)$ in the proof of such questions, and verify that $F(x)$ satisfies Rolle's theorem, draw a conclusion from Rolle's theorem. Common methods for constructing auxiliary functions include:

(1) Original function method (differential equation method)

Rewrite ξ or x_0 in the conclusion as x, write the formula in a form that is easy to remove the sign of the first derivative (that is the form that is easy to integrate), remove the sign of the first derivative (*i.e.*, make an integral), transposition, make one end of the equation "0", the other end is the newly created auxiliary function $F(x)$ (for simplicity, the integration constant is taken as 0). [3]

(2) Constant k-value method

This method is often used in propositions where the constant part can be separated. Generally let the constant part be K and make constant deformation, make one end of the above equation an algebraic equation formed by a and $f(a)$, an algebraic expression formed by b and $f(b)$ at the other end. Analyze whether the expression about the endpoint is symmetric or rotationally symmetric, if a (or b) is changed to x, change the corresponding function $f(a)$ (or $f(b)$) to $f(x)$, then the endpoint expression after replacing the variable is the auxiliary function. [4]

Example 7 Let $f(x)\in C[a,b]\bigcap D(a,b), f(a)=0, a>0$, prove that there is $\xi\in(a,b)$, bring $f(\xi)=\frac{b-\xi}{b}f'(\xi)$.

Analyse: In the above formula, ξ is rewritten as x, then $f(x)=\frac{b-x}{b}f'(x)\Rightarrow\frac{f'(x)}{f(x)}=\frac{b}{b-x}$, integrated on both sides $\ln f(x)=-a\ln(b-x)+C\Rightarrow(b-x)^a f(x)=C$.

Proof: Introducing auxiliary functions $F(x)=(b-x)^a f(x)$, then $F(a)=F(b)=0$, then $F(x)$ satisfies Rolle's theorem on $[a,b]$, so there is a point $\xi\in(a,b)$, bring $F'(\xi)=0$, that is $f(\xi)=\frac{b-\xi}{b}f'(\xi)$.

Example 8 Let $f(x)\in C[a,b]\bigcap D(a,b), f(a)=0, a>0$, prove that there is $\xi\in(a,b)$, bring $\frac{bf(b)-af(a)}{b-a}=f(\xi)+\xi f'(\xi)$. [1]

Analyse: Order $k=\frac{bf(b)-af(a)}{b-a}\Rightarrow bf(b)-kb=af(a)-ka$, then it has with rotation symmetry, take $F(x)=xf(x)-kx$.

Proof: Introducing auxiliary functions $F(x)=xf(x)-kx=xf(x)-\frac{bf(b)-af(a)}{b-a}x$, then

$F(b)-F(a)=bf(b)-\frac{bf(b)-af(a)}{b-a}b-af(a)+\frac{bf(b)-af(a)}{b-a}a=0$

Then $F(x)$ satisfies Rolle's theorem on $[a,b]$, so there is a point $\xi\in(a,b)$, bring $F'(\xi)=0$, that is $\frac{bf(b)-af(a)}{b-a}=f(\xi)+\xi f'(\xi)$.

Example 9 Let the function $f(x)$ be second derivative on $\left[0,\frac{1}{2}\right]$, and $f(0)=f'(0)$, $f\left(\frac{1}{2}\right)=0$, prove that there is $\xi\in\left(0,\frac{1}{2}\right)$, bring $f''(\xi)=\frac{3f'(\xi)}{1-2\xi}$.

Analyse: $f''(\xi)(1-2\xi)-2f'(\xi)=f'(\xi)$, order $\xi=x$ then $f''(x)(1-2x)-2f'(x)=f'(x)$

$\left[f'(x)(1-2x)\right]'=f'(x)$, then integrated on both sides, so $f'(x)(1-2x)=f(x)+C$, order C=0, transposition $f'(x)(1-2x)-f(x)=0$.

Proof: Introducing auxiliary functions $F(x)=f'(x)(1-2x)-f(x)$, obviously $F(x)\in C\left[0,\frac{1}{2}\right]\bigcap D\left(0,\frac{1}{2}\right)$, and $F(0)=f'(0)(1-0)-f(0)=0$, $F\left(\frac{1}{2}\right)=f'\left(\frac{1}{2}\right)\left(1-2\cdot\frac{1}{2}\right)-f\left(\frac{1}{2}\right)=0$, Then $F(x)$ satisfies Rolle's theorem on $\left[0,\frac{1}{2}\right]$, so there is a point $\xi\in\left(0,\frac{1}{2}\right)$, bring $F'(\xi)=0$, that is $f''(\xi)(1-2\xi)-3f'(\xi)=0$, that is $f''(\xi)=\frac{3f'(\xi)}{1-2\xi}$.

3. Summary

After analyzing and proving the above types of differential mean value theorem, the analysis process needs to look for appropriate auxiliary functions according to different types of questions, starting from conclusions or conditions, to meet the corresponding theorem conditions, and pay attention to logic in the process of proving. I hope to help students have a more intuitive and clear understanding of the problem type of differential mean value theorem, so that students can flexibly apply it in later higher mathematics learning [5].

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