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On the Quaternion Representation for Octonion Generalization of Lorentz Boosts

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Abstract

The paper proposes an approach to the generalization of Lorentz transformations for the real Euclidean spacetime of double dimensions. The approach is based on symmetry considerations. It provides: a) Generalized additive decomposition of a linear operator into self-adjoint (symmetric) and skewsymmetric parts; b) development of the apparatus of non-associative octonions due to the double generalization of the cross vector product for three arguments and eight-dimensional space; c) quaternion representation of Lorentz transformations as a linear combination of spatial rotation and one more orthogonal transformation; d) analytical solution of the eigenvector/eigenvalue problem for the composition of Lorentz boosts, in order to extend the quaternion record of Lorentz boost composition to the octonionic case. In general, our studies are consistent with those of Tevian Dray and Corinne A. Manogue, but are limited to using only ordinary quaternions and octonions.

Keywords

Lorentz boosts, quaternions, octonions, cross vector product ,eigenvectors, eigenvalues, Hermitian operator decomposition

1. Introduction

William Rowan Hamilton interpreted quaternions as vectors of real four-dimensional spacetime [1]. Since Hamilton's quaternions in a unique way generalize as octonions, the idea of an eight-dimensional spacetime arises. In order to meaningfully interpret octonions [2] as an eight-dimensional spacetime, it is useful to generalize Lorentz transformation for them.

As noted in [3, 4], the use of quaternions is still underestimated in the physics community, as there is no generally accepted way of using quaternions to represent Lorentz transformations, and many different authors such as L. Silberstein, P. A. M. Dirac, P. Rastol, P. R. Girard, A. A. Ungar, S. I. Mocanu, S. De Leo, J. F. Barrett and followers use their own quite distinct methods. Perhaps the elegant quaternion representation of Lorentz boosts, discovered by the American chemist D. B. Sweetser [5], is just what is needed. At least, it would be appropriate to mention Sweetser's representation in the last paragraph of the section «Quaternions in vector symbolic» [6]. In this section, Erwin Madelung refers to the class of Lorentz transformations the orthogonal transformation $bub, (b,b)=1$ of the complex plane spanned by vector b in the quaternion or octonion space of vectors u, so that the novice reader does not confuse Lorentz transformations with orthogonal ones.

It is clear that the generalization of spacetime to the case of double dimensions is not unique. Therefore, simply writing Lorentz transformations in terms of quaternions is not enough. And the trick is to use additional symmetry considerations, which are irreplaceable in such problems.

2. Double Generalization of the Cross Vector Product

At present, mainstream conventional calculations in terms of quaternions and, especially, octonions, are more of an art than a beaten path [2, 7]. To transform the art into the routine calculations, it seems useful to first generalize the additive decomposition of the operator into self-adjoint and skewsymmetric parts, and then generalize the cross vector product in a double sense.

Let A be a linear operator in some vector space. Let the operation « $^+$ » be the linear conversion that doesn't change the operator A if it is performed twice: $A^{++} = A$. Then the conventional additive decomposition [8] expresses a linear operator A as the sum of its symmetric part $(+)$ and its skewsymmetric part $(-)$:

$$\begin{matrix} A = (+) + (-) \\ A^+ = (+) - (-) \end{matrix} \Leftrightarrow \begin{matrix} (+) = \dfrac{A + A^+}{2} \\ (-) = \dfrac{A - A^+}{2} \end{matrix}.$$

The trivial property is expressed by the relationships: $(+)^+ = (+)$ and $(-)^+ = -(-)$, which specify the change or preservation of the sign of treated parts under the action of the operation « $^+$ ». In the space of vectors u and v with a real inner product (u, v), a typical example of the operation « $^+$ » is the Hermitian conjugation, which is defined by transferring from one part of the inner product to another: A^+: $(Au, v) = (u, A^+ v)$. Let's agree to interpret the operation « $^+$ » only in this sense.

For several operations of Hermitian conjugation type, which constitute an Abelian group of second-order elements, the linear operator A is analogously decomposed into a sum of mixed symmetric-skewsymmetric operators that under the action of each operation either do not change or change the sign. In the case of two commuting operations « $^+$ », and « * », the symmetric-skewsymmetric expansion of the operator is expressed by the decompositions into symmetric–skewsymmetric parts $\begin{pmatrix}+\\+\end{pmatrix}$, $\begin{pmatrix}+\\-\end{pmatrix}$, $\begin{pmatrix}-\\+\end{pmatrix}$ and $\begin{pmatrix}-\\-\end{pmatrix}$ in the form:

$$\begin{matrix} A = \begin{pmatrix}+\\+\end{pmatrix} + \begin{pmatrix}-\\+\end{pmatrix} + \begin{pmatrix}+\\-\end{pmatrix} + \begin{pmatrix}-\\-\end{pmatrix} \\ A^+ = \begin{pmatrix}+\\+\end{pmatrix} - \begin{pmatrix}-\\+\end{pmatrix} + \begin{pmatrix}+\\-\end{pmatrix} - \begin{pmatrix}-\\-\end{pmatrix} \\ A^* = \begin{pmatrix}+\\+\end{pmatrix} + \begin{pmatrix}-\\+\end{pmatrix} - \begin{pmatrix}+\\-\end{pmatrix} - \begin{pmatrix}-\\-\end{pmatrix} \\ A^{+*} = \begin{pmatrix}+\\+\end{pmatrix} - \begin{pmatrix}-\\+\end{pmatrix} - \begin{pmatrix}+\\-\end{pmatrix} + \begin{pmatrix}-\\-\end{pmatrix} \end{matrix} \Leftrightarrow \begin{matrix} \begin{pmatrix}+\\+\end{pmatrix} = \dfrac{A + A^+ + A^* + A^{+*}}{4} \\ \begin{pmatrix}-\\+\end{pmatrix} = \dfrac{A - A^+ + A^* - A^{+*}}{4} \\ \begin{pmatrix}+\\-\end{pmatrix} = \dfrac{A + A^+ - A^* - A^{+*}}{4} \\ \begin{pmatrix}-\\-\end{pmatrix} = \dfrac{A - A^+ - A^* + A^{+*}}{4} \end{matrix}, \tag{1}$$

where the upper signs in the two-component designations of the considered parts encode the action of the operation « $^+$ », and the lower ones correspond to the operation « * ». So that: $\begin{pmatrix}+\\+\end{pmatrix}^+ = \begin{pmatrix}+\\+\end{pmatrix}^* = \begin{pmatrix}+\\+\end{pmatrix}^{+*} = \begin{pmatrix}+\\+\end{pmatrix}$, $\begin{pmatrix}+\\-\end{pmatrix}^+ = \begin{pmatrix}+\\-\end{pmatrix}$, $\begin{pmatrix}+\\-\end{pmatrix}^* = \begin{pmatrix}+\\-\end{pmatrix}^{+*} = -\begin{pmatrix}+\\-\end{pmatrix}$, $\begin{pmatrix}-\\+\end{pmatrix}^+ = \begin{pmatrix}-\\+\end{pmatrix}^{+*} = -\begin{pmatrix}-\\+\end{pmatrix}$, $\begin{pmatrix}-\\+\end{pmatrix}^* = \begin{pmatrix}-\\+\end{pmatrix}$ and $\begin{pmatrix}-\\-\end{pmatrix}^+ = \begin{pmatrix}-\\-\end{pmatrix}^* = -\begin{pmatrix}-\\-\end{pmatrix}$, $\begin{pmatrix}-\\-\end{pmatrix}^{+*} = \begin{pmatrix}-\\-\end{pmatrix}$.

Let's treat the product $Au \equiv (u_1 \overline{u}) u_2$ of three octonions u_1, u and u_2, with a conjugated central variable octonion $\overline{u} = 2(u, i_0) i_0 - u$, where parentheses denote the *inner* product of vectors, and i_0 is the multiplicative unit in the octonion algebra.

Let operation « * » be an operation of inversion of multiplicative order, which is carried out by conjugation of u together with both parameters u_1 and u_2 in combination with common conjugation: A^*: $A^* u = \overline{(\overline{u}_1 \overline{\overline{u}}) \overline{u}_2}$.

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Then the results of operations « $^+$ » and « * » are summarized in Table 1

Table 1. Operations with the Product of Three Octonions

	«»	« $^+$ »	« * »
«»	$Au \equiv u_1\bar{u}u_2$	$A^+u = (u_2\bar{u})u_1$	$A^*u = u_2(\bar{u}u_1)$
« $^+$ »	$A^+u = u_2\bar{u}u_1$	$A^{++}u = Au$	$A^{+^*}u = u_1(\bar{u}u_2)$
« * »	$A^*u = u_2(\bar{u}u_1)$	$A^{*+}u = u_1(\bar{u}u_2)$	$A^{**}u = Au$

Table 1 describes the results of actions of two operations « $^+$ », « * ». The table is symmetric and contains the same diagonal elements. This indicates that the operations under consideration form an Abelian group of the second-order elements.

It is a good exercise to create a tables similar to Table 1 for $Au = (u_1 u)u_2$, $Au = u_1(\bar{u}u_2)$ and others.

Substituting the values of the elements of Table 1 in (1) we immediately get dual expressions:

$$\{u_1, u, u_2\} \equiv \begin{pmatrix} + \\ + \end{pmatrix} u = \frac{(u_1\bar{u})u_2 + (u_2\bar{u})u_1}{2} \equiv \frac{u_1(\bar{u}u_2) + u_2(\bar{u}u_1)}{2}, \quad (2)$$

$$[u_1, u, u_2] \equiv \begin{pmatrix} - \\ - \end{pmatrix} u = \frac{(u_1\bar{u})u_2 - u_2(\bar{u}u_1)}{2} \equiv \frac{u_1(\bar{u}u_2) - (u_2\bar{u})u_1}{2}, \quad (3)$$

$$\langle u_1, u, u_2 \rangle \equiv \begin{pmatrix} - \\ + \end{pmatrix} u = \frac{(u_1\bar{u})u_2 - u_1(\bar{u}u_2)}{2} \equiv \frac{u_2(\bar{u}u_1) - (u_2\bar{u})u_1}{2}, \quad (4)$$

$$\begin{pmatrix} + \\ - \end{pmatrix} u = 0, \quad (5)$$

where $\{u_1, u, u_2\}$ denotes a triple anticommutator, $[u_1, u, u_2]$ denotes a triple commutator that is the customary notation for a cross vector product generalized to the case of three arguments, and $\langle u_1, u, u_2 \rangle$ denotes well known associator [9].

These anticommutator, commutator, and associator are mutually orthogonal and define the following expansions Au, A^+u, A^*u, and $A^{*+}u$:

$$Au \equiv (u_1\bar{u})u_2 = \begin{pmatrix} + \\ + \end{pmatrix} + \begin{pmatrix} - \\ + \end{pmatrix} + \begin{pmatrix} + \\ - \end{pmatrix} + \begin{pmatrix} - \\ - \end{pmatrix} = \{u_1, u, u_2\} + [u_1, u, u_2] + \langle u_1, u, u_2 \rangle, \quad (6)$$

$$A^+u \equiv (u_2\bar{u})u_1 = \begin{pmatrix} + \\ + \end{pmatrix} - \begin{pmatrix} - \\ + \end{pmatrix} + \begin{pmatrix} + \\ - \end{pmatrix} - \begin{pmatrix} - \\ - \end{pmatrix} = \{u_1, u, u_2\} - [u_1, u, u_2] - \langle u_1, u, u_2 \rangle, \quad (7)$$

$$A^*u \equiv u_2(\bar{u}u_1) = \begin{pmatrix} + \\ + \end{pmatrix} + \begin{pmatrix} - \\ + \end{pmatrix} - \begin{pmatrix} + \\ - \end{pmatrix} - \begin{pmatrix} - \\ - \end{pmatrix} = \{u_1, u, u_2\} - [u_1, u, u_2] + \langle u_1, u, u_2 \rangle, \quad (8)$$

$$A^{*+}u \equiv u_1(\bar{u}u_2) = \begin{pmatrix} + \\ + \end{pmatrix} - \begin{pmatrix} - \\ + \end{pmatrix} - \begin{pmatrix} + \\ - \end{pmatrix} + \begin{pmatrix} - \\ - \end{pmatrix} = \{u_1, u, u_2\} + [u_1, u, u_2] - \langle u_1, u, u_2 \rangle, \quad (9)$$

The formulae (3) express the result of the cross product generalizing in octonions to the case of three arguments, which is derived from symmetry considerations.

The generalized cross product $[u_1, u, u_2]$ of vectors u_1, u, u_2 is converted to the conventional cross product $[u_1, u_2]$ of vectors u_1, u_2 when replacing the central argument u with the multiplicative unit i_0:

$$[u_1, u_2] = [u_1, i_0, u_2] \equiv \frac{u_1u_2 - u_2u_1}{2}. \quad (10)$$

The specified definition (3) provides a generalization of the usual properties of the cross product, namely, zeroing for the same arguments:

$$[u_1, u_2, u_1] = [u_1, u_2, u_1] = [u_1, u_2, u_2] = 0 \quad (11)$$

and orthogonality of $[u_1,u,u_2]$ to each of the arguments:

$$([u_1,u,u_2],u_1)=([u_1,u,u_2],u)=([u_1,u,u_2],u_2)=0\,. \tag{12}$$

As it follows from (11) the cross product of vectors changes the sign when rearranging two arguments:

$$-[u_1,u,u_2]=[u,u_1,u_2]=[u_2,u,u_1]=[u_1,u_2,u]\,. \tag{13}$$

and similar formulae can be derived from (12).

Although (3) is easy to obtain from symmetry considerations [10], it took dozens of years to guess exactly this one of a kind dual definition. So, in [11] the possible dimensionalities of the generalized cross product were specified. In [12], the formula (6) was almost derived, but in a disassembled form and without the conjugation of the central factor. In [13], the formula for $[u_1,u,u_2]$ was proposed with a different arrangement of brackets. In [14], it was noted that in the case of the location of brackets in accordance with [13], there are not one, but two alternative definitions from which the best cannot be chosen.

In (11)-(13) relations, the triple cross vector product $[u_1,u,u_2]$ can be replaced by an associator $\langle u_1,u,u_2\rangle$. However, for quaternion arguments, the associator vanishes. In common case, the associator vanishes if at least one of the arguments is proportional to the multiplicative unit i_0.

The triple anticommutator $\{u_1,u,u_2\}$ is expressed in a simple way in terms of inner products:

$$\{u_1,u,u_2\}=(u_1,u),u_2+(u_2,u)u_1-(u_1,u_2)u\,. \tag{14}$$

The listed and other properties of the triple anticommutator, the triple vector product and the associator are described in detail in [10].

3. Lorentz Transformations in Terms of Quaternions

Lorentz transformation $\mathscr{L}$ is defined as a linear homogeneous transformation of the spacetime vectors *u*, *v* that preserves the real inner product $(u,\bar{v})$ of one vector u by another *conjugated* vector $\bar{v}\equiv 2(v,i_0)i_0-v$:

$$\left(\mathscr{L}u,\overline{\mathscr{L}v}\right)=(u,\bar{v})\,, \tag{15}$$

where i_0 is the *unit* vector of unit length $\sqrt{(i_0,i_0)}\equiv 1$ along the time axis.

For brevity, only one option of $\pm\mathscr{L}u$ and $\pm\overline{\mathscr{L}u}$ is treated.

The formula (15) is equivalent to the expression for the inverse operator $\mathscr{L}^{-1}$:

$$\mathscr{L}^{-1}u=\overline{\mathscr{L}^{+}\bar{u}}\equiv\mathscr{L}^{"-1"}u\,, \tag{16}$$

where the cross denotes the Hermitian conjugation in the quaternion space with the real-valued inner product of vectors u, v: $\mathscr{L}^{+}:(\mathscr{L}u,v)=\left(\mathscr{L}u,\mathscr{L}^{+}v\right)$ and « $^{"-1"}$ » denotes the linear operation over operators.

For what follows, it is important that the inversion of the operator $\mathscr{L}$ into $\mathscr{L}^{-1}$ is the linear operation over the set of Lorentz transformations, since it is a superposition of two linear operations.

It is well known that Lorentz transformation $\mathscr{L}$ is expressed by the superposition of the rotation V and *self-adjoint* Lorentz boost L, so that: $\mathscr{L}=VL$, where $L:(Lu,v)=\left(u,L^{+}v\right)$.

A quaternion record of Lorentz boost L was found in [5] and accomplished in [15]. It turns out that Lorentz boost is decomposed into a linear combination of rotation either $\dfrac{\bar{a}ua}{\cosh\theta}$ or $\dfrac{au\bar{a}}{\cosh\theta}$ and orthogonal multiplicative transformation either $n\bar{u}$ or $\bar{u}n$, expressing in twofold ways by the formulae:

$$Lu=\bar{a}ua-\sinh\theta\cdot n\bar{u}=L^{+}u=au\bar{a}-\sinh\theta\cdot\bar{u}n\,, \tag{17}$$

where $a=i_0\cdot\mathrm{ch}\dfrac{\theta}{2}+n\cdot\mathrm{sh}\dfrac{\theta}{2}$, i_0 is the multiplicative unity (identity), n is the unit vector along the speed, θ is the rapidity: $\mathrm{th}\,\theta=\dfrac{\mathrm{v}}{\mathrm{c}}$, v is the speed magnitude, c is the speed of light.

It is noteworthy that in [16] a halved expression $au\bar{a}$ was obtained due to the excessive dimension of the space.

Expressions (17) of Lorentz boost through a pair of orthogonal transformations justify the consideration of real space-time with a real Euclidean inner product.

The quaternion expressions (17) of Lorentz boost, as well as the rotation expression $Vu = bu\bar{b}$, $(b,b)=1$, due to the alternativeness of octonion multiplication, do not require parentheses as do not depend on the multiplication order and retain their meaning in octonions. At first glance, this is quite sufficient for the eight-dimensional generalization of Lorentz transformations VL as a superposition of rotation V and Lorentz boost L. This may be true, but other options should also be considered for comparison.

In four-dimensional space, both the rotation V and Lorentz boost L are the elementary transformations that modify some two-dimensional plane and do not change the orthogonal vectors in another two-dimensional plane.

In both quaternions and octonions, Lorentz boost (17) describes the stretching by a certain number of times of one vector and the contraction by the same number of times of another vector that belong to the same complex plane while preserving purely spatial vectors that are orthogonal to this complex plane. The transformation of the rotation $Vu = bu\bar{b}$, $(b,b)=1$ in octonions is more complicated than Lorentz boost, because it modifies the rest six-dimensional subspace of pure spatial vectors, while maintaining the complex plane defined by the rotational axis. Maybe, it is better to represent the rotation as a superposition of a pair of reflections? Anyway, additional considerations are required.

4. Eigenvector Quartet for Composition of Lorenz Boosts

General Lorentz transformations in the form of a composition VL of rotation V and boost L are subdivided into boost-like transformations with a quartet of basis vectors, and the rest ones, referring to as rotation-like. The most interesting is that the composition L_1L_2 of Lorentz boosts L_1 and L_2 is always a boost-like transformation [15].

The eigenvectors for the composition L_1L_2 of Lorentz boosts together with the corresponding eigenvalues are listed in Table 2.

Table 2. Eigenvectors for the composition L_1L_2 of Lorentz boosts L_1, L_2

Notation	Eigenvector	Eigenvalue
c_0	$i_0 - d\big\vert_{\zeta=\exp(\chi)}$	$\exp(\chi)$
c_1	$i_0 - d\big\vert_{\zeta=\exp(-\chi)}$	$\exp(-\chi)$
c_2	$i_0 - n_1\dfrac{\coth\frac{\theta_1}{2}+(n_1,n_2)\coth\frac{\theta_2}{2}}{1-(n_1,n_2)^2} + n_2\dfrac{\coth\frac{\theta_2}{2}+(n_1,n_2)\coth\frac{\theta_1}{2}}{1-(n_1,n_2)^2}$	1
c_3	$[n_1,n_2]$	1

In Table 1, n_1 and n_2 are the unit spatial vectors along the considered intersecting velocities, such that $(n_1,n_1)=(n_2,n_2)=1$ and $(n_1,i_0)=(n_2,i_0)=0$. The cross vector product $[n_1,n_2]$ is directed along the Wigner rotational axis [17], so that $[n_1,n_2]=\nu\sqrt{1-(n_1,n_2)^2}$. The spatial part of the eigenvectors c_0 and c_1 depends on the eigenvalue ζ and, up to the sign, coincides with the unit vector $d|_\zeta$ that is defined as a function of eigenvalue ζ in the form [15]:

$$d\big|_\zeta = \frac{n_1\sqrt{\zeta}\sinh\frac{\theta_1}{2} + n_2\sinh\frac{\theta_2}{2}}{\sqrt{\zeta}\cosh\frac{\theta_1}{2} - \cosh\frac{\theta_2}{2}}. \quad (18)$$

The spatial parts $-d|_{\zeta=\exp(\chi)}$ and $-d|_{\zeta=\exp(-\chi)}$ of the eigenvectors c_0 and c_1 are obtained by substituting in (18) the values ζ by $\exp(\chi)$ and $\exp(-\chi)$, respectively. The scalar parameter χ is defined in accordance with well-known cosine rule:

$$\cosh\frac{\chi}{2}=\cosh\frac{\theta_1}{2}\cosh\frac{\theta_2}{2}+(n_1,n_2)\sinh\frac{\theta_1}{2}\sinh\frac{\theta_2}{2} \tag{19}$$

and the scalar parameters θ_1 and θ_2 are the rapidities, such that the velocities v_1, v_2 divided by scalar speed of light c are expressed as $\mathrm{v}_1/\mathrm{c}=n_1\tanh\theta_1$, $\mathrm{v}_2/\mathrm{c}=n_2\tanh\theta_2$. Note that (19) refers to the half hyperbolic angles $\chi/2$, $\theta_1/2$ and $\theta_2/2$, while the well-known velocity addition is expressed via holistic hyperbolic angles θ, θ_1 and θ_2 [18, 19].

The eigenvectors from Table 2 form the basis of the considered space $\mathbb{R}^4$. Their pairwise pseudoscalar inner products (c_i,c_k), $i,k=0,1,2,3$, accounting for commutativity, are given by the formulae (20).

$$\begin{gathered}
(c_0,\bar{c}_0)=(c_0,\bar{c}_2)=(c_0,\bar{c}_3)=(c_1,\bar{c}_1)=(c_1,\bar{c}_2)=(c_1,\bar{c}_3)=(c_2,\bar{c}_3)=0,\\
(c_0,\bar{c}_1)=-2\frac{\sinh^2\frac{\chi}{2}}{\cosh^2\frac{\theta_1}{2}+\cosh^2\frac{\theta_2}{2}-2\cosh\frac{\chi}{2}\cosh\frac{\theta_1}{2}\cosh\frac{\theta_2}{2}},\\
(c_2,\bar{c}_2)=1-\frac{\coth^2\frac{\theta_1}{2}+\coth^2\frac{\theta_2}{2}+2(n_1,n_2)\coth\frac{\theta_1}{2}\coth\frac{\theta_2}{2}}{1-(n_1,n_2)^2},\\
(c_3,\bar{c}_3)=(n_1,n_2)^2-1.
\end{gathered} \tag{20}$$

5. Expressing Lorentz boost composition in terms of eigenvectors

Let's continue consideration of the composition of Lorentz boosts in terms of quaternions.

According to formulae (20), an arbitrary four-dimensional vector u can be easily represented as a linear combination of eigenvectors of Lorentz boost composition, and then, using Table 2, get Lorentz boost composition L_1L_2 itself and related expressions $(L_1L_2)^+$, $(L_1L_2)^{-1}$ and $(L_1L_2)^{-1+}$ in the form:

$$\begin{gathered}
L_1L_2u=\frac{c_0\exp(\chi)(u,\bar{c}_1)+c_1\exp(-\chi)(u,\bar{c}_0)}{(c_0,\bar{c}_1)}+c_2\frac{(u,\bar{c}_2)}{(c_2,\bar{c}_2)}+c_3\frac{(u,\bar{c}_3)}{(c_3,\bar{c}_3)}\\
(L_1L_2)^+u=\frac{\bar{c}_0\exp(-\chi)(u,c_1)+\bar{c}_1\exp(\chi)(u,c_0)}{(c_0,\bar{c}_1)}+\bar{c}_2\frac{(u,c_2)}{(c_2,\bar{c}_2)}+\bar{c}_3\frac{(u,c_3)}{(c_3,\bar{c}_3)}=L_2L_1u\\
(L_1L_2)^{-1}u=\frac{c_0\exp(-\chi)(u,\bar{c}_1)+c_1\exp(\chi)(u,\bar{c}_0)}{(c_0,\bar{c}_1)}+c_2\frac{(u,\bar{c}_2)}{(c_2,\bar{c}_2)}+c_3\frac{(u,\bar{c}_3)}{(c_3,\bar{c}_3)}=L_2^{-1}L_1^{-1}u\\
(L_1L_2)^{-1+}u=\frac{\bar{c}_0\exp(\chi)(u,c_1)+\bar{c}_1\exp(-\chi)(u,c_0)}{(c_0,\bar{c}_1)}+\bar{c}_2\frac{(u,c_2)}{(c_2,\bar{c}_2)}+\bar{c}_3\frac{(u,c_3)}{(c_3,\bar{c}_3)}=L_1^{-1}L_2^{-1}u=\overline{L_1L_2\bar{u}}
\end{gathered} \tag{21}$$

The initial composition of two Lorenz boosts is expressed by the formula, which is written out at the top. The formula is attractive because it describes the boost composition L_1L_2 through its invariants, namely, constant pseudoscalar products and eigenvectors that preserve their directions. In this case, one eigenvector is stretched, the other is compressed by the same factor, and the rest two remain the same. Thus, the boost composition is interpreted as simply as a single Lorentz boost.

It is also attractive that any transformation (21) is reversed by simply changing the sign of the parameter γ.

The disadvantage of the formula for L_1L_2 is that it does not use quaternion multiplication. Meanwhile, according to (17) boost L and with it the general Lorentz transformation VL is elegantly expressed through quaternion multiplication as a linear combination of two orthogonal transformations, first of which is a rotation. So, in the general case, Lorentz transformations VL and, in particular, the composition of Lorentz boosts L_1L_2 and also the accompanying $(L_1L_2)^+$, $(L_1L_2)^{-1}\equiv(L_1L_2)^{"-1"}$, $(L_1L_2)^{-1+}$ are represented as:

$$
\begin{aligned}
L_1L_2u &= b\bar{a}ua\bar{b} - \sinh\theta\cdot bn\bar{u}\bar{b} = bau\bar{a}\bar{b} - \sinh\theta\cdot b\bar{u}n\bar{b},\\
(L_1L_2)^{+}u &= L_2L_1u = \bar{a}\bar{b}uba - \sinh\theta\cdot n\bar{b}\bar{u}b = a\bar{b}ub\bar{a} - \sinh\theta\cdot \bar{b}\bar{u}bn,\\
(L_1L_2)^{-1}u &= L_2{}^{-1}L_1{}^{-1}u = a\bar{b}ub\bar{a} + \sinh\theta\cdot n\bar{b}\bar{u}b = \bar{a}\bar{b}uba + \sinh\theta\cdot \bar{b}\bar{u}bn,\\
(L_1L_2)^{+-1}u &= L_1{}^{-1}L_2{}^{-1}u = \overline{L_1L_2\bar{u}} = b\bar{a}ua\bar{b} + \sinh\theta\cdot b\bar{u}n\bar{b} = bau\bar{a}\bar{b} + \sinh\theta\cdot bn\bar{u}\bar{b},
\end{aligned}
\tag{22}
$$

where $a = i_0\cdot \mathrm{ch}\dfrac{\theta}{2} + n\cdot \mathrm{sh}\dfrac{\theta}{2}$, unit vector n along the speed, the rapidity θ: $\mathrm{th}\,\theta = \dfrac{\mathrm{v}}{\mathrm{c}}$, the speed magnitude v, $b = i_0\cdot\cos\dfrac{\varphi}{2} + \nu\cdot\sin\dfrac{\varphi}{2}$, unit vector ν along the rotational axis, rotational angle φ, matching the composition L_1L_2, are regarded as unknown parameters.

Solution of the problem «head on» by directly comparing the formulae for L_1L_2 in (21) and (22) and additional formulae for expressing the parameters for L_1L_2 in (21) via the parameters for L_1L_2 in (22) does not use the parameter γ and leads to cumbersome calculations. Probably, the operator expansion (1) into symmetric-skewsymmetric parts will be useful for obtaining the expected concise expressions.

Let's decompose L_1L_2 in terms of a pair of operations « $^{+}$ » and « $^{"-1"}$ » (formula (16)).

Let's agree to use in (1) square brackets instead of round ones to indicate that we are talking about the expansion of not of an arbitrary operator A, but about a specific expansion of L_1L_2 and operations « $^{+}$ » and « $^{"-1"}$ ».

Then we'll get:

$$
\begin{aligned}
L_1L_2u &= \begin{bmatrix}+\\+\end{bmatrix}u + \begin{bmatrix}+\\-\end{bmatrix}u + \begin{bmatrix}-\\+\end{bmatrix}u + \begin{bmatrix}-\\-\end{bmatrix}u\\
(L_1L_2)^{+}u &= \begin{bmatrix}+\\+\end{bmatrix}u + \begin{bmatrix}+\\-\end{bmatrix}u - \begin{bmatrix}-\\+\end{bmatrix}u - \begin{bmatrix}-\\-\end{bmatrix}u\\
(L_1L_2)^{-1}u &= \begin{bmatrix}+\\+\end{bmatrix}u - \begin{bmatrix}+\\-\end{bmatrix}u + \begin{bmatrix}-\\+\end{bmatrix}u - \begin{bmatrix}-\\-\end{bmatrix}u\\
(L_1L_2)^{-1+}u &= \begin{bmatrix}+\\+\end{bmatrix}u - \begin{bmatrix}+\\-\end{bmatrix}u - \begin{bmatrix}-\\+\end{bmatrix}u + \begin{bmatrix}-\\-\end{bmatrix}u
\end{aligned}
\quad\Leftrightarrow\quad
\begin{aligned}
\begin{bmatrix}+\\+\end{bmatrix}u &= \frac{L_1L_2u + (L_1L_2)^{+}u + (L_1L_2)^{-1}u + (L_1L_2)^{-1+}u}{4}\\
\begin{bmatrix}+\\-\end{bmatrix}u &= \frac{L_1L_2u + (L_1L_2)^{+}u - (L_1L_2)^{-1}u - (L_1L_2)^{-1+}u}{4}\\
\begin{bmatrix}-\\+\end{bmatrix}u &= \frac{L_1L_2u - (L_1L_2)^{+}u + (L_1L_2)^{-1}u - (L_1L_2)^{-1+}u}{4}\\
\begin{bmatrix}-\\-\end{bmatrix}u &= \frac{L_1L_2u - (L_1L_2)^{+}u - (L_1L_2)^{-1}u + (L_1L_2)^{-1+}u}{4}
\end{aligned}.
\tag{23}
$$

Since the rotation B commutes with the conjugation operation, then in difference expressions of the $L_1L_2 - (L_1L_2)^{-1+}$ type, the (22) dependence on the rotation $b\bar{a}ua\bar{b}$, $bau\bar{a}\bar{b}$, etc. is eliminated when substituting into (23). This is reassuring.

Thus, it is probably useful to “assemble” the formula for L_1L_2 in (21) into the sum of two products of quaternion triples like in (22), using eigenvectors and eigenvalues from Tab 2. Then it will be possible to see whether the record retains its meaning in octonions, or whether it will be necessary to clarify the order of quaternion multiplication in triplets. As a result, the necessary considerations will probably appear for the best generalization of Lorentz transformations and the concept of double-dimensional spacetime.

6. Conclusions

General Lorentz transformations and, in particular, Lorentz boost composition L_1L_2 are elegantly represented by the additive combination of the pair of orthogonal transformations, one of which is rotation. A feature of just the composition of a pair of Lorentz boosts is that it always has an eigenvector quartet. We have established that the composition L_1L_2 of Lorentz boosts L_1, L_2 in terms of stretching/shrinking of eigenvectors is described in the same way as a single Lorentz boost. These considerations allow to suppose that there is an elegant quaternion representation for Lorentz boost composition L_1L_2, which essentially uses the parameter γ, ensuring the transition to the inverse transformation $(L_1L_2)^{-1}$ when it's sign is reversed.

The only obstacle to immediately solve the problem is cumbersome calculations. Calculations in quaternions by themselves provide conciseness of calculations. To further reduce the complexity of calculations, it seems useful to use an additive decomposition of the composition of Lorentz boosts into symmetric-skewsymmetric parts. It is possible that this will lead to success, just as it turned out to be useful for the double generalization of the cross vector product. For guaranteed eliminate computational errors, it would be useful if someone tried to generalize the composition of Lorentz boosts to the case of octonions along with the author.

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