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Galerkin Residual Correction for Fourth Order BVP

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Abstract

This article uses residual correction procedure for improving the Galerkin approximate solutions to higher order boundary value problem (BVP). The residual function of a differential equation is found from the approximate solution of a BVP and setting it as nonhomogeneous term we get the error differential equation. We exploit Bernstein and Bernoulli polynomials as basis functions to solve the two differential equations, namely, original and its error equations, by Galerkin technique subject to the corresponding boundary conditions. Linear and nonlinear problems of fourth order BVPs are considered to verify the proposed method. The resulting numerical solutions are compared with the analytic solutions as well as the results of other approaches those have been reported in the literature. This method is also applied to sixth order BVPs. The comparison reveals that the current procedure is more accurate.

Keywords

Galerkin Method, Linear and Nonlinear BVP, Bernstein Polynomials, Bernoulli Polynomials, Residual Correction

1. Introduction

Higher order boundary value problem arises in various application of science and engineering such as oxygen diffusion in cells [1], heat and mass transfer within porous particle [2], hydrodynamics and hydromagnetic stability [3], boundary layer theory, control and optimization theory etc. Fourth order BVP are extensively used to model viscoelastic flows [4], simulate deformations of elastic beams on elastic bearings [5] and in the study of plate deflection theory [6,7]. The analytical solution of most of the higher order ordinary differential equations with complicated boundary condition is not easy to determine and as a result numerous types of numerical methods have been introduced by many prominent authors. The numerical approaches of fourth-order BVPs are well-documented in [6, 8-15]. Many efficient numerical methods have been proposed and employed to approximate solution of such BVPs. Examples includes finite difference method (FDM), which is comparatively easy but to achieve high accuracy, it requires a large number of parameters. Finite difference methods generate numerical solutions at grid points only while Galerkin weighted residual method finds approximate results at any point in the domain of a problem. Islam and Hossain [16, 17] applied Galerkin technique to solve higher order BVPs utilizing Legendre and Bernoulli polynomials. On the other hand, Adomian decomposition method (ADM) and its modifications [18-20], homotopy perturbation method(HPM), homotopy analysis method (HAM) [21, 22], and the differential transform method (DTM) [23, 24] have been applied for the solution of higher order BVPs. Very recently, Adak and Mandal [25] have solved Euler–Bernoulli beam equation with Neumann boundary condition by FDM. Sixth-order boundary-value problems (BVPs) can model problems in astrophysics and the stable layers bounded narrow convecting layers, which possibly surround A-type stars [26-28]. Agarwal [29] presented the theorems stating the conditions for the existence and uniqueness of solutions of sixth order boundary value problems.

Continuing interest in research work on BVPs, Oliveira [30] applied residual correction to solve linear BVP using collocation method for the original differential equation with finite difference method to solve the error differential equation. Celik [31] studied the same equation applying Chebyshev series method. But, they have limited their discussions for only linear second order differential equations. Siddiqi and Twizell [32] used the sextic spline to generate the solution of sixth order boundary value problem. Boutayeb and Twizell [28] proposed a class of numerical methods for the solution of special nonlinear boundary value problems. El-Gamel et al. [33] used Sinc-Galerkin scheme for the solutions of sixth order boundary value problems. Quintic spline and septic spline were studied for different step sizes in Siddiqi [34, 35]. Khan and Sultana [36] considered second order and fourth order parametric spline technique to solve two point BVPs. Uniform Har wavelets is proposed for the numerikal solution of sixh order BVPs by Haq et al. [37]. To the best of our knowledge, none has attempted using Galerkin technique with residual correction to solve higher order BVPs utilizing Bernstein and Bernoulli polynomials.

However, in this paper, we give a short introduction of Bernstein and Bernoulli polynomials and their properties in Section 2. The formulation of Galerkin weighted residual correction for linear and nonlinear BVPs of fourth order is discussed in Section 3. Numerical examples are considered in section 4, and the results are compared with the solutions obtained previously by the several methods.

2. Some special polynomials

(a) Bernstein polynomials: The Bernstein polynomials [17, 38-41] of n-th degree have the following typical form over the interval $[a,b]$:

$$P_{i,n}(t)=\binom{n}{i}\frac{(t-a)^i(b-t)^{n-i}}{(b-a)^n}, a\le t\le b, i=0,1,2,...,n$$

For instance, these are the first five Bernstein polynomials of degree 4 over the domain $[0,1]$:

$$P_0(t)=(1-t)^4, P_1(t)=4(1-t)^3t, P_2(t)=6(1-t)^2t^2, P_3(t)=4(1-t)t^3, P_4(t)=t^4$$

It's worth noting that the polynomials have the following properties:

(i) $P_{i,n}(t)=0$ if $i<0$ or $i>n$

(ii) $\sum_{i=0}^{n}P_{i,n}(t)=1$

(iii) $P_{i,n}(a)=P_{i,n}(b)=0,\ i=1,2,...,n-1$

In the Galerkin technique to solve a BVP, Bernstein polynomials are utilized in the trail functions that fulfill the associated homogeneous form of the essential boundary conditions.

(b) Bernoulli Polynomials: Over the domain $[0,1]$, the Bernoulli polynomials [42-46] of degree n can be defined implicitly by

$$P_n(t)=\sum_{j=0}^{n}\binom{n}{j}b_jt^{n-j}$$

where b_j are Bernoulli numbers given by

$$b_0=1 \text{ and } b_j=-\int_0^1 P_j(t)dt,\ k\ge 1.$$

Bernoulli polynomials can be written in a different way

$$P_0(t)=1$$
$$P_m(t)=\sum_{n=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(t+k)^m, m\ge 1$$

The first few Bernoulli polynomials are given bellow:

$$P_0(t)=1,\quad P_1(t)=t-\frac{1}{2},\quad P_2(t)=-t+t^2+\frac{1}{6},\quad P_3(t)=\frac{t}{2}-\frac{3t^2}{2}+t^3,\quad P_4(t)=t^2-2t^3+t^4-\frac{1}{30}$$

The characteristics of nth order Bernoulli polynomial are

$$P_n(0) = (-1)^n P_n(1), \text{ for } n = 0,1,2,...$$

We employ the Bernoulli polynomials in the following form to meet the homogeneity criterion of the non-derivative boundary conditions.

$$\hat{P}_m(t) = \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (t+k)^m - \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m, m \geq 1$$

Now we have $\hat{P}_n(0) = \hat{P}_n(1) = 0$ for $n = 1,2,3,...$and

$$\hat{P}_1(t) = t\,, \quad \hat{P}_2(t) = -t + t^2\,, \quad \hat{P}_3(t) = \frac{t}{2} - \frac{3t^2}{2} + t^3\,, \quad \hat{P}_4(t) = t^2 - 2t^3 + t^4\,, \,...$$

3. Residual Correction Method

Since Oliveira [30] first introduced residual correction with finite difference method to solve 2$^{\text{nd}}$ order linear BVP using collocation method. Later, Celik [31] applied the same equation with Chebyshev series method. However, in this section, we propose the formulation of residual correction method for fourth order BVP. For this, we consider a linear fourth order boundary value issue in its most generic form

$$L[u] = u^{iv}(t) + p_3(t)u'''(t) + p_2(t)u''(t) + p_1(t)u'(t) + p_0(t)u(t) = f(t), \; a \leq t \leq b \tag{1}$$

with linearly independent boundary conditions

$$\sum_{k=0}^{3} \left[\alpha_{ik} u^k(a) + \beta_{ik} u^k(b) \right] = \gamma_i, \; i = 0,1,2,3 \tag{2}$$

Let $u(t)$ be the exact solution of (1) and (2), and $\tilde{u}(t)$ be a trial solution in Galerkin method. By applying the boundary conditions, the coefficients of the trial solution are determined to meet the conventional Galerkin equations. The residual function $R(t)$ of the operator equation is

$$R(t) = L[\tilde{u}(t)] - f(t), \; a \leq t \leq b \tag{3}$$

The error function $E(t) = u(t) - \tilde{u}(t)$ was studied by Olivera [30]. Since L is a linear operator, we have

$$L[E(t)] = L[u(t)] - L[\tilde{u}(t)] = -R(t) \tag{4}$$

The Galerkin approach [47] with Bernstein and Bernoulli polynomials were used to produce $\tilde{u}(t)$, and the error differential equation (4) was solved using the same procedure. We develop the matrix formulation for fourth-order BVP to exemplify the method, and then expand our notion to solve higher-order BVP. A fourth-order linear differential equation is considered as follows:

$$-\frac{d^2}{dt^2}\left(p(t) \frac{d^2 u}{dt^2} \right) + q(t)u = r(t), \; a \leq x \leq b \tag{5a}$$

with non-derivative boundary conditions,

$$u(a) = a_1, u(b) = a_2 \tag{5b}$$

and derivative boundary conditions

$$u''(a) = c_1, u''(b) = c_2 \tag{5c}$$

where a_1, a_2, c_1, c_2 are constants and $p(t)$, $q(t)$, $r(t)$ are continuous functions. The solution of the differential equation (5) is approximated as

$$\tilde{u}(t) = \theta_0(t) + \sum_{i=1}^{n} a_i P_i(t), n \geq 1 \tag{6}$$

Here, $\theta_0(t)$ is met by the basic boundary conditions in (5b) and where $P_i(t)$, $i = 1,2,3,\ldots,n$ are Bernstein and

Bernoulli polynomials and $P_i(0) = P_i(1) = 0$. The Galerkin weighted residual equations may be found by substituting (6) into equation (5a).

$$\int_a^b \left[-\frac{d^2}{dt^2}\left(p(t)\frac{d^2\tilde{u}}{dt^2} \right) + q(t)\tilde{u} - r(t) \right] P_i(t)\,dt = 0,\ i = 1,2,\ldots,n \tag{7}$$

Simplifying, we obtain

$$\begin{aligned} &\sum_{j=1}^{n}\left[\int_a^b \left[p(t)\frac{d^2P_i}{dt^2}\frac{d^2P_j}{dt^2} + q(t)P_i(t)P_j(t) \right] dt \right] a_j \\ &= \int_a^b r(t)P_i(t)\,dt - \int_a^b \left[p(t)\frac{d^2\theta_0}{dt^2}\frac{d^2P_i}{dt^2} + q(t)\theta_0(t)P_i(t) \right] dt \\ &+ c_2\, p(b)\frac{dP_i}{dt}(b) - c_1\, p(a)\frac{dP_i}{dt}(a) \end{aligned} \tag{8}$$

Or, in matrix notations,

$$\sum_{j=1}^{n} K_{ij}a_j = F_i,\ i = 1,2,\ldots,n \tag{9a}$$

where

$$K_{ij} = \int_a^b \left[p(t)\frac{d^2P_i}{dt^2}\frac{d^2P_j}{dt^2} + q(t)P_i(t)P_j(t) \right] dt \tag{9b}$$

$$\begin{aligned} F_i &= \int_a^b r(t)P_i(t)\,dt - \int_a^b \left[p(t)\frac{d^2\theta_0}{dt^2}\frac{d^2P_i}{dt^2} + q(t)\theta_0(t)P_i(t) \right] dt \\ &+ c_2\, p(b)\frac{dP_i}{dt}(b) - c_1\, p(a)\frac{dP_i}{dt}(a) \end{aligned} \tag{9c}$$

The values of the parameters a_i's are obtained by solving the system $(9a)$. Substituting into (6) gives the approximate solution $\tilde{y}(t)$ of the desired BVP (5).

The error differential equation is thus

$$-\frac{d^2}{dt^2}\left(p(t)\frac{d^2E}{dt^2} \right) + q(t)E = -R(t), ?a \le t \le b \tag{10a}$$

with boundary conditions,

$$E(a) = 0, E(b) = 0 \tag{10b}$$

$$E''(a) = c_1', E''(b) = c_2' \tag{10c}$$

where $R(t)$ is the residual function derived from the differential equation (5) has an explicit form

$$R(t) = -\frac{d^2}{dt^2}\left(p(t)\frac{d^2\tilde{u}}{dt^2} \right) + q(t)\tilde{u} - r(t) \tag{11}$$

and c_1', c_2' are some constants defined by $c_1' = c_1 - \tilde{u}''(a)$, $c_2' = c_2 - \tilde{u}''(b)$. We solve equation (10) by Galerkin method using the same polynomials as above. The solution of the error differential equation is approximated as

$$\tilde{E}(t) = \sum_{i=1}^{n} a_i P_i(t),\ n \ge 1 \tag{12}$$

We obtain the values of the parameters a_i by solving a system analogous to $(9a)$ and then substitute them into (12)

to get the approximate solution $\tilde{E}(t)$ of the desired BVP (10). Summing the approximate solutions $\tilde{u}(t)$ and $\tilde{E}(t)$ we get an improved approximation $\tilde{u}_1(t)=\tilde{u}(t)+\tilde{E}(t)$.

4. Convergence Analysis

Let $X=C^m[a,b]$ be the vector space of m times differentiable real valued functions on the interval $[a,b]$. Define L^2 inner product on X as $\langle f,g\rangle=\int_a^b r(t)f(t)g(t)dt$, for some sufficiently smooth weight function $r(t)$. Then the L^2 norm of a function f is defined by $\|f\|=\sqrt{\int_a^b r(t)f^2(t)dt}$. X is a Hilbert space. Assume that $B=\{\varphi_k \mid k=1,2,\cdots\}$ is a Schauder basis of X. Let us begin with a finite dimensional subspace, called an approximation space X^N of X spanned by $\{\varphi_k \mid k=1,2,\cdots,N\}$ such that each φ_k satisfies the boundary conditions (2). The Gelerkin weighted residual equation is given by $\langle R(\tilde{y}(t),t),\varphi_k\rangle=\int_a^b r(t)R(\tilde{y}(t),t)\varphi_k dt=0$, where $\tilde{y}(t)=\sum_1^N c_k\varphi_k$, is equivalent to finding $\tilde{y}(t)\in X^N$ such that $\langle R(\tilde{y}(t),t),\varphi(t)\rangle=\int_a^b r(t)R(\tilde{y}(t),t)\varphi(t)dt=0$ for all $\varphi\in X^N$. That is $R(\tilde{y}(t),t)$ is orthogonal to the subspace X^N. In particular, $R(\tilde{y}(t),t)$ is orthogonal to the generating functions $\{\varphi_k \mid k=1,2,\cdots,N\}$. Observe that, if the dimension of the subspace X^N is increased to infinity, the residual $R(\tilde{y}(t),t)$ is orthogonal to each member of the Schauder basis $B=\{\varphi_k \mid k=1,2,\cdots\}$. That is $R(\tilde{y}(x),x)$ is orthogonal to any function in X. A function orthogonal to any other function is necessarily the zero function.

Therefore, $\lim_{n\to\infty}R(\tilde{y}(t),t)=\lim_{n\to\infty}\left(L[y(t)]-L[\tilde{y}(t)]\right)=0$.

Hence $y(t)=\lim_{n\to\infty}\tilde{y}(t)=y(t)$. Gelerkin weighted residual method uses the approximation $\tilde{y}(t)=\sum_1^N c_k\varphi_k$ and determines the constants c_k's from the relation $\langle R(\tilde{y}(t),t),\varphi(t)\rangle=0$. Hence, the approximation converges to the exact solution.

As the error function $E(t)$ induced from the approximation of the exact solution by $\tilde{y}(t)=\sum_1^N c_k\varphi_k$ is again estimated from the Galerkin method, the estimated error is guaranteed to converge to the actual error. That is why two fold applications of the Galerkin methods gives better approximation.

5. Numerical Examples

In this section, we use the residual approach described in the preceding section to a variety of issues from the literature to illustrate its efficiency and applicability. We consider four and two numerical examples for fourth and sixth order BVPs, respectively, including nonlinear differential equations.

Example 1: We consider the linear Euler–Bernoulli beam BVP [25]

$$\frac{d^4u}{dt^4}=t, 0\le t\le 1 \tag{13a}$$

$$u(0)=u(1)=0,\ u''(0)=0,\ u'(1)=0 \tag{13b}$$

The exact solution is

$$u(t)=\frac{t}{120}-\frac{t^3}{60}+\frac{t^5}{120}$$

The error differential equation related to the above BVP is

$$\frac{d^4E}{dt^4}=-R(t), 0\le t\le 1 \tag{14a}$$

$$E(0)=E(1)=0,\quad E''(0)=-\tilde{u}''(0), E'(1)=-\tilde{u}'(1) \tag{14b}$$

For 3 polynomials we have $R(t)=\frac{1}{2}-t$

Table 1. Observed Maximum Absolute Error (MAE) for example 1

Number of polynomials	Present method		Reference results [25]
	Bernstein Polynomials	Bernoulli polynomials	
2	7.35×10^{-5}	7.35×10^{-5}	
3	1.20×10^{-17}	2.79×10^{-18}	7.56×10^{-4} *with* $h=\frac{1}{4}$
4	6.99×10^{-29}	1.06×10^{-31}	5.46×10^{-4} *with* $h=\frac{1}{8}$
5	2.18×10^{-29}	2.80×10^{-31}	

Adak and Mandal [25] have employed a finite difference approach with a consistent step size of $h=\frac{1}{4}$ and $h=\frac{1}{8}$ to numerically solve this BVP, and they have reported maximum errors of 7.56×10^{-4} for $h=\frac{1}{4}$ and 5.46×10^{-4} for $h=\frac{1}{8}$. That is, the step size was reduced, but the accuracy order remained same. Our answer closely resembles the precise solution for this sort of problem. Table 1 shows that for only 5 polynomials, the greatest error is approximately zero, with a value of 2.18×10^{-29} for Bernstein polynomials and 2.80×10^{-31} for Bernoulli polynomials and we get the approximate polynomial in both cases as

$$\tilde{u}_1(t)=0.0083333t-0.016667t^3+0.0083333t^5.$$

Example 2: We consider the linear BVP [16, 48-51]

$$\frac{d^4u}{dt^4}+tu=-\left(8+7t+t^3\right)\exp(t), 0\le t\le 1 \tag{15a}$$

$$u(0)=u(1)=0, u''(0)=0, u''(1)=-4\exp(1) \tag{15b}$$

with exact solution: $u(t)=t(1-t)\exp(t)$.

The error differential equation related to the above BVP is

$$\frac{d^4E}{dt^4}+tE=-R(t),\quad 0\le t\le 1 \tag{16a}$$

$$E(0)=E(1)=0, E''(0)=-\tilde{u}''(0), E''(1)=-4\exp(1)-\tilde{u}''(1) \tag{16b}$$

$R(t)$ is the appropriate residual function derived from the approximate solution $\tilde{u}(t)$ of equation (15) and for 5 polynomials which is

$$R(t)=-8.864-8.4768t-24.0511t^2-0.0011t^3-0.48935t^4-0.36933t^5-0.07064t^6$$
$$-0.069t^7+(8+7t+t^3)e^t$$

Table 2. Observed Maximum Absolute Error (MAE) for example 2

Number of polynomials	Present method		Previous results obtained in Ref. [16]	Reference results
	Bernstein Polynomials	Bernoulli polynomials		
3	1.55×10^{-4}	1.55×10^{-4}	2.46×10^{-3}	
5	2.73×10^{-7}	2.73×10^{-7}	5.94×10^{-6}	[48]: 3.72×10^{-11}
7	2.81×10^{-10}	2.81×10^{-10}	9.01×10^{-9}	[49]: 5.01×10^{-11} [50]: 4.13×10^{-7}
9	1.86×10^{-13}	1.81×10^{-13}	7.13×10^{-12}	[51]: 5.37×10^{-6}
11	1.39×10^{-16}	9.67×10^{-15}	-	

The maximum absolute errors for different number of polynomials are displayed in Table 2. This problem was also solved by Islam and Hossain [16] using Galerkin method with various types of polynomials. Fourth order method of first kind was used by Loghmani and Alavizadeh [48] had an error of 3.72×10^{-11} and fourth degree B-spline function with 84 hat functions in Ramadan et al. [49] with error 5.01×10^{-11}. Again the quartic spline solution [50] with $h = \frac{1}{256}$ and the quintic spline procedure [51] with step size $h = \frac{1}{64}$ provided maximum errors 4.13×10^{-7} and 5.37×10^{-6}, respectively. It is obvious from the results in Table 2 that our method is more accurate than the above methods, since we have got a higher accuracy with maximum error 1.39×10^{-16} using only 11 Bernstein polynomials and 9.67×10^{-15} with only 11 Bernoulli polynomials.

Example 3: In this example, we consider a nonlinear BVP [8]

$$\frac{d^4u}{dt^4} = e^{-t}u^2,\ 0 \le t \le 1 \tag{17a}$$

$$u(0) = 1, u(1) = e \tag{17b}$$

$$u''(0) = 1, u''(1) = e \tag{17c}$$

with the analytic solution of this BVP is $u(t) = e^t$.

We assume that $\tilde{u}(t) = \theta_0(t) + \sum_{i=1}^{n} a_i P_i(t), n \ge 1$

Here, $\theta_0(t) = t(e-1)+1$ that is satisfied by the boundary conditions in $(19b)$ and $P_i(0) = P_i(1) = 0$, $i = 1,2,3,\dots,n$. The error differential equation related to equation (19) is obtained by replacing $u(t)$ by $\tilde{u}(t) + E(t)$.

$$\frac{d^4E}{dt^4} - e^{-t}E^2 - 2e^{-t}\tilde{u}E = -R(t), \quad 0 \le t \le 1 \tag{18a}$$

$$E(0) = 0, E(1) = 0 \tag{18b}$$

$$E''(0) = 1 - \tilde{u}''(0),\ E''(1) = e - \tilde{u}''(1) \tag{18c}$$

For 5 polynomials, the residual function is

$$R(t) = 1.0212 + 0.83t + 0.83t^2 + (-1 - 2t - 2t^2 - 1.33t^3 - 0.66t^4 - 0.26t^5 - 0.08t^6 - 0.025t^7)e^{-t}$$
$$+(-0.0064t^8 - 0.0013t^9 - 0.00024t^{10} - 0.0000323t^{11} - 5.36\times10^{-6}t^{12})e^{-t}$$

Table 3. Observed Maximum Absolute Error (MAE) for example 3

Number of polynomials	Present method		Reference results in [8]
	Bernstein Polynomials	Bernoulli polynomials	
2	5.19×10^{-6}	5.18×10^{-6}	MADM: 7.61×10^{-4} DMG: 8.26×10^{-8}
3	1.40×10^{-7}	1.41×10^{-7}	
5	1.25×10^{-10}	1.25×10^{-10}	
7	6.48×10^{-14}	6.48×10^{-14}	

This problem was also solved by Singh et al. [8] using Modified Adomian decomposition method (MADM) and Decomposition method with Green's function (DMG). Table 3 compares the maximum absolute error obtained by the present method, MADM and DMG. MADM gave maximum error only 7.61×10^{-4} and our scheme gives 1.40×10^{-7} for only 3 polynomials. For DMG, the error was obtained as 8.26×10^{-8} whereas 1.25×10^{-10} is obtained using only 5 polynomials. We see that, a high accuracy with MAE 6.48×10^{-14} is obtained by present method using 7 polynomials.

Example 4: We consider a fourth order nonlinear BVP [8, 52]

$$\frac{d^4u}{dt^4}=g(t)+u^2,\quad 0\le t\le 1 \tag{19a}$$

$$u(0)=0,u(1)=0 \tag{19b}$$

$$u'(0)=0,u'(1)=1 \tag{19c}$$

where, $g(t)=-t^{10}+4t^9-4t^8-4t^7+8t^6-4t^4+120t-48$,

and the analytical solution of this BVP is: $u(t)=t^5-2t^4+2t^2$

We assume that $\tilde{u}(t)=\theta_0(t)+\sum_{i=1}^{n}a_iP_i(t), n\ge 1$

Here, $\theta_0(t)=t$ that is satisfied by the boundary conditions in $(19b)$ and $P_i(0)=P_i(1)=0$, $i=1,2,3,\dots,n$. The error differential equation related to equation (19) is obtained by replacing $u(t)$ by $\tilde{u}(t)+E(t)$.

$$\frac{d^4E}{dt^4}-2\tilde{u}E-E^2=-R(t),0\le t\le 1 \tag{20a}$$

$$E(0)=0, E(1)=0 \tag{20b}$$

$$E'(0)=-\tilde{u}'(0), E'(1)=-\tilde{u}'(1) \tag{20c}$$

The residual function for three polynomials is shown here

$$R(t)=59.99-120t+0.00045t^3-2.24t^4+9.99t^5-14.49t^6+5.99t^7+3.75t^8-4t^9+t^{10}.$$

Table 4. Observed Maximum Absolute Error (MAE) for example 4

Number of polynomials	Present method		Reference results
	Bernstein Polynomials	Bernoulli polynomials	
2	8.86×10^{-3}	8.86×10^{-3}	[8] DMG: 5.67×10^{-12} [8] MADM: 6.36×10^{-7} [52]: 6.66×10^{-15}
3	1.09×10^{-6}	1.09×10^{-6}	
4	1.10×10^{-39}	7.34×10^{-40}	
5	2.93×10^{-39}	3.67×10^{-40}	

An MAE of 6.36×10^{-7}[8] was reported for the three term truncated solution by modified Adomain decomposition method (MADM), whereas Singh et al. [8] reported an error of 5.67×10^{-12} for the three term truncated solution by DMG. It is obvious that DMG was a superior technique to MADM. Variational iteration method was applied in [52] and obtained an MAE 6.66×10^{-15}. Table 4 shows that residual correction produces a numerical solution with great accuracy as 1.10×10^{-39} for just 4 Bernstein polynomials and 7.34×10^{-40} for 4 Bernoulli polynomials, and the approximate polynomial is thus $\tilde{u}_1(t)=2t^2-2t^4+t^5$ which is similar to exact solutions.

Example 5: Consider the sixth order linear BVP [34-37, 53]

$$\frac{d^6u}{dt^6}-u=-6e^{-t}, 0\le t\le 1 \tag{21a}$$

$$u(0)=1,u(1)=0,u'(0)=0,u'(1)=-e,u''(0)=-1,u''(1)=-2e \tag{21b}$$

The analytic solution of this BVP is $u(t)=(1-t)e^t$

We assume that $\tilde{u}(t)=\theta_0(t)+\sum_{i=1}^{n}a_iP_i(t), n\ge 1$

Here, $\theta_0(t)=1-t$ that is satisfied by the non-derivative boundary conditions and $P_i(0)=P_i(1)=0$, $i=1,2,3,\dots,n$. The error differential equation related to equation (21) is obtained by replacing $u(t)$ by $\tilde{u}(t)+E(t)$.

$$\frac{d^6E}{dt^6} - E = -R(t), \quad 0 \le t \le 1 \tag{22a}$$

$$\begin{aligned} &E(0) = 0, E(1) = 0, E'(0) = -\tilde{u}'(0), E'(1) = -e - \tilde{u}'(1), \\ &E''(0) = -1 - \tilde{u}''(0), E''(1) = -2e - \tilde{u}''(1) \end{aligned} \tag{22b}$$

For 10 polynomials the residual function is

$$\begin{aligned} R(t) = &-3528.6 + 62659.0t - 331955.0t^2 + 736533.0t^3 - 724177.0t^4 + 260400.0t^5 + \\ &4.8994t^6 - 12.432t^7 + 16.466t^8 - 12.178t^9 + 4.7895t^{10} - 0.78284t^{11} + 6e^t \end{aligned}$$

Table 5. Observed Maximum Absolute Error (MAE) for example 5

Number of polynomials	Present method		Previous results obtained in Ref. [17]	Reference results
	Bernstein Polynomials	Bernoulli polynomials		
10	1.45×10^{-13}	1.45×10^{-13}	7.87×10^{-12}	[34]: 1.23×10^{-9} [35]: 7.04×10^{-9} [36]: 9.39×10^{-11} [37]: 1.09×10^{-11} [53]: 2.55×10^{-9}
11	2.33×10^{-15}	2.33×10^{-15}	1.18×10^{-13}	
12	8.69×10^{-17}	8.69×10^{-17}	5.95×10^{-14}	
13	5.02×10^{-17}	5.02×10^{-17}	8.77×10^{-15}	

Islam and Hossain [17] solved this BVP using Galerkin method but they did not apply residual correction procedure and got MAE 8.77×10^{-15} for 13 polynomials. Quintic spline [34] and septic spline [35] methods for step size $h = \frac{1}{64}$ gave MAE 1.23×10^{-9} and 7.04×10^{-9} respectively whereas 9.39×10^{-11} was obtained for $h = \frac{1}{64}$ and parameters $(p, q, s) = (0, \frac{1}{4}, \frac{1}{2})$ in fourth order parametric spline technique. Considering 1024 Har coefficients Haq et al. showed MAE 1.09×10^{-11} and with $h = \frac{1}{32}$, $(\alpha, \beta, \gamma, \delta) = (\frac{1}{5040}, \frac{6}{504}, \frac{1250}{5040}, \frac{2418}{5040})$, MAE was obtained as 2.55×10^{-9} using nonpolynomial spline in Akram [53]. Our residual scheme gave very high accuracy with MAE 8.69×10^{-17} for only 12 polynomials.

Example 6: Let us consider the sixth order nonlinear BVP [17, 18]

$$\frac{d^6u}{dt^6} = u^2 e^t, \quad 0 \le t \le 1 \tag{23a}$$

$$u(0) = 1, u(1) = e^{-1}, u'(0) = -1, u'(1) = -e^{-1}, u''(0) = 1, u''(1) = e^{-1} \tag{23b}$$

whose analytic solution is: $u(t) = e^{-t}$

We assume that $\tilde{u}(t) = \theta_0(t) + \sum_{i=1}^{n} a_i P_i(t), \, n \ge 1$

Here, $\theta_0(t) = 1 - t(1 - e^{-1})$ satisfies the essential boundary conditions in (23b) and $P_i(0) = P_i(1) = 0$, $i = 1,2,3, \ldots, n$. The error differential equation related to equation (23) is obtained by replacing $u(t)$ by $\tilde{u}(t) + E(t)$.

$$\frac{d^6E}{dt^6} - 2e^t \tilde{u} E - e^t E^2 = -R(t), \quad 0 \le t \le 1 \tag{24a}$$

$$\begin{aligned} &E(0) = 0, E(1) = 0, E'(0) = -1 - \tilde{u}'(0), E'(1) = -e^{-1} - \tilde{u}'(1), \\ &E''(0) = 1 - \tilde{u}''(0), E''(1) = e^{-1} - \tilde{u}''(1) \end{aligned} \tag{24b}$$

where residual function for 5 polynomials is

$$R(t) = 0.62 + \begin{pmatrix} -1 + 1.99t - 1.99t^2 + 1.33t^3 - 0.66t^4 + 0.26t^5 - 0.08t^6 + 0.02t^7 \\ -0.0052t^8 + 0.0009t^9 - 0.00013t^{10} + 0.000013t^{11} - 0.000000745t^{12} \end{pmatrix} e^t$$

Wazwaz [18] also solved this nonlinear BVP utilizing a modified form of Adomain decomposition method but he received maximum error only 6.19×10^{-6}. Form the Table 6, we observe that the order of accuracy did not increase with the polynomial number in [17]. But we obtain 9.13×10^{-8} for only 5 polynomials by the present method which is lower than 3.11×10^{-7} for 10 polynomials in [17]. Our solution is very accurate with an error 4.62×10^{-15} and 7.72×10^{-17} in case of 10 Bernstein polynomials and 10 Bernoulli polynomials, respectively.

Table 6. Observed Maximum Absolute Error (MAE) for example 6

Number of polynomials	Present method		Previous results obtained in Ref. [17]	Results in [18]
	Bernstein Polynomials	Bernoulli polynomials		
5	9.13×10^{-8}	9.13×10^{-8}		
6	1.11×10^{-9}	1.11×10^{-9}	2.14×10^{-7}	
8	2.76×10^{-13}	2.76×10^{-13}	2.24×10^{-7}	6.19×10^{-6}
10	4.62×10^{-15}	7.72×10^{-17}	3.11×10^{-7}	

6. Conclusions

We have used the residual correction of Galerkin technique to solve linear and nonlinear BVPs using Bernstein and Bernoulli polynomials and focused on the performance of the procedure. For this, we have applied the formulation on fourth order BVPs and compared approximate solutions with the analytical/ numerical solutions available in the references, and found that they are in good agreement. The results of the previous section indicate that residual correction procedure can be used to obtain better numerical solutions of boundary value problems in both linear and nonlinear cases. It is observed that double layer application Galerkin residual correction achieve much higher accuracy compared with the results obtained from single layer of the method. Sometimes, the obtained polynomial coincides with the exact solution.

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