Weighted Approximation Properties of New \((p, q)\)—Analogue of Balázs Szabados Operators

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Abstract

Korovkin-type theorems provide simple and useful tools for finding out whether a given sequence of positive linear operators, acting on some function space is an approximation processor, equivalently, converges strongly to the identity operator. These theorems exhibit a variety of test subsets of functions which guarantee that the approximation property holds on the whole space provided it holds on them. These kinds of results are called “Korovkin-type theorems” which refers to P.P. Korovkin who in 1953 discovered such a property for the functions \(1, x, x^2\) in the space \(C([0,1])\). After this discovery, several mathematicians have undertaken the program of extending Korovkin’s theorems in many ways and to several settings. Such developments delineated a theory which is nowadays referred to as Korovkin-type approximation theory. In this paper, we study weighted approximation properties of new \((p, q)\) - analogue of the Balázs-Szabados operators by using the weighted modulus of continuity and we give a Korovkin type theorem for weighted approximation.

Keywords
\((p, q)\)- analysis, moments, Bernstein operators, Balázs-Szabados operators, \((p, q)\)-Balázs-Szabados operators, weighted modulus of continuity

1. Introduction

In the year 1975, Catherine Balázs defined and studied Bernstein type rational functions as follows (see [1]),

\[
R_n(f;x) = \frac{1}{1+a_n x} \sum_{k=0}^{\infty} f\left(k \frac{b_n}{a_n}\right) \left(\frac{n}{a_n x}\right)^k \left(a_n x\right)^{n-k} \quad (n = 1, 2, \ldots),
\]

where \(f\) is a real valued and single valued function which is defined on the unbounded interval \([0, \infty)\), \(a_n\) and \(b_n\) are real numbers which are selected suitably and do not depend on \(x\). Seven years later in 1982, Catherine Balázs and J. Szabados studied together to improve the estimation in [2] by selecting suitable \(a_n\) and \(b_n\) under some restrictions for \(f(x)\).

Recently, generalizations of Balázs-Szabados operators based on the \(q\)-integers are studied by Hayatem Hamal and Pembe Sabancigil ([3]), Ogün Doğru ([4]) and Esma Yıldız Özkan ([5]). Approximation properties of the \(q\)-Balázs-Szabados complex operators are studied by Nazım I. Mahmudov in [6] and by Nurhayat Ispir and Esma Yıldız Özkan in [7].
Moreover, the fast rise of \((p,q)\)-analysis has encouraged many authors in this subject to discover different generalizations and examine their approximation properties. In the last seven years, Mohammad Mursaleen et al. introduced and studied \((p,q)\)-analogue of Bernstein operators, \((p,q)\)-analogue of Bernstein-Stancu operators, Bernstein-Kantorovich operators based on \((p,q)\)-calculus, \((p,q)\)-Lorentz polynomials on a compact disc, Bleimann-Butzer-Hahn operators defined by \((p,q)\)-integers and \((p,q)\)-analogue of two parametric Stancu-Beta operators (see [8]-[14]). \((p,q)\)-generalization of Szász-Mirakyan operators is studied by Tuncer Acar (see [15]), Kantorovich modification of \((p,q)\)-Bernstein operators is studied by Tuncer Acar and Ali Aral (see [16]). A generalization of \(q\)-Balázs-Szabados operators based on \((p,q)\)-integers is studied by Esma Yıldız Özkan and Nurhayat Ispir in [17]. Hayatem Hamal and Pembe Sabancigil introduce a new \((p,q)\)-generalization of \(q\)-Balázs-Szabados operators as follows (see [18]),

\[
R_{n,p,q}(f,x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} p^{(k-1)/2} f \left( \frac{p^{n-k} [k]_{p,q}}{b_n} \right) \left( \frac{a_{n,k}}{1+a_n x} \right)^k \prod_{j=0}^{n-1} \left( p^j - q^j \frac{a_n x}{1+a_n x} \right),
\]

where \(a_n = [n]_{p,q}^{p-1}, b_n = [n]_{p,q}^q, 0 < \beta \leq \frac{2}{3}, n \in \mathbb{N}, x \geq 0, f\) is a real-valued function defined on the unbounded interval \([0, \infty)\).

In this paper, we study weighted approximation properties of new \((p,q)\)-analogue of the Balázs-Szabados operators by using the weighted modulus of continuity and we give a Korovkin type theorem for weighted approximation.

Before giving the main results for the mentioned operators above, we present some important notations and some basic definitions of \((p,q)\)-analysis. For any two non-negative numbers \(p, q\) and a non-negative integer \(n\), the \((p,q)\)-integer of the number \(n\) is defined as follows:

\[
\begin{align*}
[n]_{p,q} &= p^{n-1} + p^{n-2} q + p^{n-3} q^2 + \ldots + pq^{n-2} + q^{n-1}, & \text{if } p \neq q \\ &= np^{n-1}, & \text{if } p = q \neq 1 \\ &= n, & \text{if } p = q = 1
\end{align*}
\]

The \((p,q)\)-factorial is defined by \([n]_{p,q} != \prod_{k=1}^{n} [k]_{p,q}, \) for \(n \geq 1\) and particularly we have \([0]_{p,q} != 1.\)

\((p,q)\)-binomial coefficient is defined by

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q} !}{[k]_{p,q} ! [n-k]_{p,q} !}, 0 \leq k \leq n,
\]

and the formula of \((p,q)\)-binomial expansion is defined by

\[
(ax + by)^n = \sum_{k=0}^{n} \binom{n}{k}_{p,q} \frac{a^{n-k} b^k}{q^k} x^{n-k} y^k = (ax + by) \left( \frac{p}{q} \left( p^2 a + q^2 b \right) \right) \ldots \left( p^{n-1} a x + q^{n-1} b y \right).
\]

2. Main Results

Firstly we consider the following three spaces:

\[
B_2 \left[ 0, \infty \right) = \left\{ f : [0, \infty) \rightarrow \mathbb{R} : \| f(x) \| \leq M_f (1 + x^2) \right\}, \text{ where } M_f \text{ is a constant depending on } f.
\]
The norm on the space \( C^*_{\infty} \) is shown as \( \| f(x) \|_{\infty} = \sup_{x \in [0, \infty)} \frac{f(x)}{1+x^2} \).

The modulus of continuity of \( f \) on a closed and bounded interval \([0, b], b > 0\) is defined as follows:

\[
\omega_b(f, \delta) = \sup_{|x| < \delta \cap [0, b]} \left| f(x) - f(x) \right|
\]

It is obvious that for a function \( f \in C^*_{\infty} \), the modulus of continuity \( \omega_b(f, \delta) \) tends to zero as \( \delta \to 0 \).

**Definition 1** [18] Let \( 0 < q < p \leq 1 \), we introduce a new \((p, q)\)-analogue of Balázs-Szabados operators by

\[
R_{n,p,q}(f, x) = \frac{1}{p^n(b-x)^{1/2}} \sum_{k=0}^{n} \left( \frac{p^{k-1}}{p-q} \right) \left( \frac{a_k}{1+a_k x^{1/2}} \right)^{1/2} \prod_{j=0}^{n-k-1} \left( \frac{p^{j+1}}{p^{j+1}-1} \right) x^{1/2},
\]

where \( a_n = \left[ \frac{n}{p} \right] \), \( b_n = \left[ \frac{n}{q} \right] \), \( 0 < \beta \leq \frac{2}{3} \), \( n \in \mathbb{N} \), \( x > 0 \), \( f \) is a real-valued function which is defined on the unbounded interval \([0, \infty)\).

In the following theorem we give the rate of convergence for the new \((p, q)\)-analogue of Balázs-Szabados operators, \( R_{n,p,q}(f, x) \).

**Theorem 1** ([19], [20]). Let \( f \in C^*_{\infty} \), \( 0 < q < p \leq 1 \) and \( \omega_{b_{n-1}}(f, \delta) \) be the modulus of continuity on \([0, b+1] \subset [0, \infty)\), where \( b > 0 \). Then for every \( n \in \mathbb{N} \), we have

\[
\left\| R_{n,p,q}(f, x) - f(x) \right\|_{[0,b]} \leq L + 2\omega_{b_{n-1}}(f, \delta)
\]

where \( L \) is a positive constant.

**Proof.** For \( x \in [0, b] \) and \( t > b+1 \), since \( t-x > 1 \) we have

\[
\left| f(t) - f(x) \right| \leq M_f \left( 2 + x^2 + t^2 \right) \leq M_f \left( 2(t-x)^2 + 3x^2(t-x)^2 + 2(t-x)^2 \right)
\]

\[
\leq M_f \left( 4 + 3x^2 \right)(t-x)^2 \leq 4M_f \left( 1+b^2 \right)(t-x)^2.
\]

(1)

For \( x \in [0, b], t < b+1 \), we have

\[
\left| f(t) - f(x) \right| \leq \omega_{b_{n-1}}(f, |t-x|) \leq \left( 1 + \frac{|t-x|}{\delta} \right) \omega_{b_{n-1}}(f, \delta).
\]

(2)

So, with \( \delta > 0, x \in [0, b], t \geq 0 \) and by the inequalities (1) and (2) we may write

\[
\left| f(t) - f(x) \right| \leq 4M_f \left( 1+b^2 \right)(t-x)^2 + \left( 1 + \frac{|t-x|}{\delta} \right) \omega_{b_{n-1}}(f, \delta),
\]

by applying \( R_{n,p,q}(f, x) \) to the above inequality and by the well known Cauchy-Schwarz Inequality, we obtain

\[
\left| R_{n,p,q}(f, x) - f(x) \right| \leq 4M_f \left( 1+b^2 \right) R_{n,p,q}(t-x)^2, x + \left( 1 + \frac{1}{\delta} R_{n,p,q}(t-x)^2, x \right) \omega_{b_{n-1}}(f, \delta),
\]

\[
\left| R_{n,p,q}(f, x) - f(x) \right| \leq 4M_f \left( 1+b^2 \right) R_{n,p,q}(t-x)^2, x + \left( 1 + \frac{1}{\delta} R_{n,p,q}(t-x)^2, x \right)^{1/2} \omega_{b_{n-1}}(f, \delta).
\]
Now by using Lemma 3 in [18], we may write
\[
|R_{n,p,q}(f,x) - f(x)| \leq 4M_f \left(1 + b^2\right)D_1 (1+x)^2 + \left(1 + \frac{1}{\delta} \sqrt{D_1} (1+x)\right)\omega_{b_1}(f,\delta),
\]
where \( D_1 \) is a positive constant.

For \( x \in [0,b] \), we have the following explicit formula:
\[
|R_{n,p,q}(f,x) - f(x)| \leq 4M_f \left(1 + b^2\right)D_1 (1+x)^2 + \left(1 + \frac{1}{\delta} \sqrt{D_1} (1+b)\right)\omega_{b_1}(f,\delta).
\]

Then, by taking \( \delta = \sqrt{D_1} (1+b) \), \( L = \left(4M_f + D_1\right) \) we get the desired result.

In the following theorem, we give Korovkin’s approximation property for the new \((p,q)\)-analogue of Balázs-Szabados operators.

**Theorem 2.** Assume that \( q = q_n, p = p_n \) are sequences such that \( 0 < q_n < p_n \leq 1 \) and \( q_n \to 1 \) as \( n \to \infty \). Then for each \( f \in C_1^1 [0,\infty) \) we have \( \lim_{n \to \infty} \|R_{n,p_n,q_n}(f,x) - f(x)\|_2 = 0 \).

**Proof.** By using the Korovkin theorem for weighted approximation (see [21], [22], [23]), it is sufficient to show that
\[
\lim_{n \to \infty} \left\| R_{n,p_n,q_n}(t^m;x) - x^m \right\|_2 = 0 \text{, for } m = 0, 1, 2.
\]

Since \( R_{n,p_n,q_n}(1;x) = 1 \), (3) holds for \( m = 0 \). Now by Lemma 2 in [18], we have
\[
R_{n,p_n,q_n}(t;x) - x = \frac{x}{1 + a_{n,p_n,q_n}x} - x = - \frac{a_{n,p_n,q_n}x^2}{(1 + a_{n,p_n,q_n})}.
\]
By using triangle inequality, we get
\[
|R_{n,p_n,q_n}(t;x) - x| \leq \frac{a_{n,p_n,q_n}x^2}{(1 + a_{n,p_n,q_n})}.
\]

Now we may write
\[
\left\| R_{n,p_n,q_n}(t;x) - x \right\|_2 \leq \sup_{0 \leq \alpha \leq c} \frac{1}{1 + x^2} \left( \frac{a_{n,p_n,q_n}x^2}{(1 + a_{n,p_n,q_n})} \right) \leq a_{n,p_n,q_n} \sup_{0 \leq \alpha \leq c} \frac{x^2}{1 + x^2 (1 + a_{n,p_n,q_n})}.
\]
By taking the limit overall the last inequality, we have
\[
\lim_{n \to \infty} \left\| R_{n,p_n,q_n}(t;x) - x \right\|_2 \leq \lim_{n \to \infty} a_{n,p_n,q_n} \cdot 1 = 0.
\]

Again by using Lemma 2 in [18], we have
\[
R_{n,p_n,q_n}(t^2;x) - x^2 = \frac{p_{n-1}}{a_{n,p_n,q_n} b_{n,p_n,q_n}} \left( \frac{a_{n,p_n,q_n}x}{1 + a_{n,p_n,q_n}x} \right)^2 + \frac{q_n [n-1]_{p_n,q_n}}{a_{n,p_n,q_n} b_{n,p_n,q_n}} \left( \frac{a_{n,p_n,q_n}x}{1 + a_{n,p_n,q_n}x} \right)^2 - x^2,
\]
\[
= \frac{p_{n-1}}{a_{n,p_n,q_n} b_{n,p_n,q_n}} \left( \frac{a_{n,p_n,q_n}x}{1 + a_{n,p_n,q_n}x} \right)^2 + \frac{q_n [n-1]_{p_n,q_n}}{n_{p_n,q_n}} \left( \frac{1}{1 + a_{n,p_n,q_n}x} \right) - 1 \right) x^2.
\]
Therefore,
\[
\left\| R_{n,p_n,q_n}(t^2;x) - x^2 \right\|_2 \leq \frac{p_{n-1}}{a_{n,p_n,q_n} b_{n,p_n,q_n}} \left( \frac{a_{n,p_n,q_n}x}{1 + a_{n,p_n,q_n}x} \right)^2 + \left( 1 - \frac{q_n [n-1]_{p_n,q_n}}{n_{p_n,q_n}} \right) x^2.
\]
Then, we have
\[
\left\| R_{n,p,q} \left( r^2; x \right) - x^2 \right\| \leq \frac{p_{n,q}^{p-1}}{b_{n,p,q}} \sup_{0 \leq s < \infty} \frac{x}{(1+x^2)(1+a_{n,p,q}x)} + \sup_{0 \leq s < \infty} \frac{x^2}{(1+x^2)(1+a_{n,p,q}x)}
\]
- \frac{p_{n,q}^{p-1}}{n_{n,p,q}} \sup_{0 \leq s < \infty} \frac{x^2}{(1+x^2)(1+a_{n,p,q}x)^2}.
\]

Now by taking the limit overall the last inequality, we have
\[
\lim_{n \to \infty} \left\| R_{n,p,q} \left( r^2; x \right) - x^2 \right\| \leq \lim_{n \to \infty} \frac{p_{n,q}^{p-1}}{b_{n,p,q}} \sup_{0 \leq s < \infty} \frac{x}{(1+x^2)(1+a_{n,p,q}x)} + \lim_{n \to \infty} \sup_{0 \leq s < \infty} \frac{x^2}{(1+x^2)(1+a_{n,p,q}x)}
\]
- \lim_{n \to \infty} \frac{p_{n,q}^{p-1}}{n_{n,p,q}} \sup_{0 \leq s < \infty} \frac{x^2}{(1+x^2)(1+a_{n,p,q}x)^2}.
\]

Hence,
\[
\lim_{n \to \infty} \left\| R_{n,p,q} \left( r^2; x \right) - x^2 \right\| = 0.
\]

Now, we present the next theorem to approximate all functions in the space $C^*_{1-} [0, \infty)$. These types of results are given in [24] for locally integrable functions.

**Theorem 3.** Let $0 < q_n < p_n < 1$, $q_n \to 1$ as $n \to \infty$. Then for each $f \in C^*_{1-} [0, \infty)$ and all $\nu > 0$, we have
\[
\lim_{n \to \infty} \sup_{x \in [0, \infty)} \frac{R_{n,p,q} \left( f, x \right) - f (x)}{(1+x^2)^{\frac{1}{1+\nu}}} = 0.
\]

**Proof.** Let $x_0 > 0$. Then
\[
\sup_{x \in [0, \infty)} \frac{R_{n,p,q} \left( f; x \right) - f (x)}{(1+x^2)^{\frac{1}{1+\nu}}} = \sup_{x \leq x_0} \frac{R_{n,p,q} \left( f; x \right) - f (x)}{(1+x^2)^{\frac{1}{1+\nu}}} + \sup_{x > x_0} \frac{R_{n,p,q} \left( f; x \right) - f (x)}{(1+x^2)^{\frac{1}{1+\nu}}}
\]
\[
\leq \left\| R_{n,p,q} \left( f; x \right) - f (x) \right\|_{[0,x_0]} + \sup_{x \in [0, \infty)} \frac{R_{n,p,q} \left( (1+r^2) f; x \right) - f (x)}{(1+x^2)^{\frac{1}{1+\nu}}}
\]
\[
\leq \left\| R_{n,p,q} \left( f; x \right) - f (x) \right\|_{[0,x_0]} + \sup_{x > x_0} \frac{R_{n,p,q} \left( (1+r^2) f; x \right) - f (x)}{(1+x^2)^{\frac{1}{1+\nu}}}.
\]

Now, by definition of the norm of each function belonging to $C^*_{1-} [0, \infty)$, we have
\[
\left\| f (x) \right\| \leq \left\| f \right\| \left( 1+x^2 \right), \text{also we have} \sup_{x \geq x_0} \frac{\left\| f (x) \right\|}{\left( 1+x^2 \right)^{\nu}} \leq \left\| f \right\|_{L^\nu} \leq \left\| f \right\| \left( 1+x_0^2 \right)^{-\nu}.
\]

Let $\epsilon > 0$. We can choose $x_0$ in such a way that
\[
\left\| f \right\| \left( 1+x_0^2 \right)^{-\nu} \leq \frac{\epsilon}{3}.
\]

By Theorem 2, we get
\[
\left\| f \right\|_L^2 \lim_{n \to \infty} \frac{R_{n,p,q} \left( \left( 1 + t^2 \right) x \right)}{\left( 1 + x^2 \right)^{1/2}} = \frac{1 + x^2}{1 + x^2} \leq \left\| f \right\|_{L^2} \leq \frac{\left\| f \right\|_{L^2}}{1 + x^2} < \frac{\varepsilon}{3}.
\]

By using Theorem 1, we can see that the first term of the inequality (4) implies that
\[
\left\| R_{n,p,q} \left( f : x \right) - f \left( x \right) \right\|_{[0, \infty)} < \frac{\varepsilon}{3}, \quad \text{as } n \to \infty
\]

By taking the limit over inequality (4) and combining inequalities (5) and (6), we get the desired result.

Next, we discuss the order of approximation of the functions \( f \in C^2 \) by the operators \( R_{n,p,q} \) with the help of the following weighted modulus of continuity (see [25] and [26]).

The weighted modulus of continuity is defined by
\[
\Omega_2 \left( f \delta \right) = \sup_{0 < h < \delta, x \in (0, \infty)} \frac{|f(x + h) - f(x)|}{1 + (x + h)^2}, \quad \forall f \in C^2 \left[ 0, \infty \right).
\]

The weighted modulus of continuity and usual modulus of continuity have similar properties.

**Lemma 1** ([26]). Let \( f \in C^2 \left[ 0, \infty \right) \). Then, we have the following:
1) \( \Omega_2 \left( f \delta \right) \) is a monotonic increasing function of \( \delta \).
2) For each \( f \in C^2 \), \( \lim_{\delta \to 0^+} \Omega_2 \left( f \delta \right) \).
3) For each \( \lambda > 0 \), \( \Omega_2 \left( f \lambda \delta \right) \leq (1 + \lambda) \Omega_2 \left( f \delta \right) \).

In the following theorem we give the main convergence result which gives an expression of the approximation error with the operators \( R_{n,p,q} \) using \( \Omega_2 \).

**Theorem 4.** Let \( f \in C^2 \), \( 0 < q_n < p_n < 1 \) such that \( q_n \to 1 \) as \( n \to \infty \). Then we have the following inequality
\[
\sup_{x \in [0, \infty]} \left| R_{n,p,q} \left( f : x \right) - f \left( x \right) \right| \leq A_1 \Omega_2 \left( f \delta \right),
\]
where, \( A_1 = 2 \left( 1 + A_1 + A_2 \right) > 0 \) and \( D_1 > 0 \).

**Proof.** It is known that \( R_{n,p,q} \left( 1, x \right) = 1 \), by monotonicity of \( R_{n,p,q} \), we may write
\[
\left| R_{n,p,q} \left( f : x \right) - f \left( x \right) \right| \leq R_{n,p,q} \left( \left| f \left( x \right) - f \left( x \right) \right|, x \right),
\]
now, by using the definition of \( \Omega_2 \left( f \delta \right) \) and (3) in the previous lemma we have
\[
\left| f \left( t \right) - f \left( x \right) \right| \leq \left( 1 + \left( x + |t - x| \right)^2 \right) \Omega_2 \left( f \delta \right).
\]

By using linearity and positivity properties of the operators \( R_{n,p,q} \), we obtain
\[
\left| R_{n,p,q} \left( f : x \right) - f \left( x \right) \right| \leq 2 \left( 1 + x^2 \right) \left( 1 + \left( t - x \right)^2 \right) \left( 1 + \frac{|t - x|}{\delta} \right) \Omega_2 \left( f \delta \right).
\]

Applying the Cauchy-Schwarz inequality on the second term of the last inequality, we get
On the other hand, in [18], by using (12) and (13) in Lemma 4, we have the following inequalities

\[
\left(1 + R_{n,p_x,q_x}(t-x)^2,x\right) \leq A_1 (1+x^2), \quad A_1 > 0.
\]

\[
\left(1 + R_{n,p_x,q_x}(t-x)^4,x\right) \leq \left(1 + x^2\right)^2, \quad \text{where } D_2 > 0.
\]

Since \( \lim_{n \to \infty} \frac{1}{b_n} = 0 \), there exists a positive constant \( A_2 \) such that

\[
\left(R_{n,p_x,q_x}(t-x)^4,x\right) \leq \left(1 + x^2\right)^2, \quad A_2 > 0.
\]

For \( 0 < q_n < p_n < 1 \), by substituting (8), (9) and (10) into the inequality (7), we can write

\[
\left|R_{n,p_x,q_x}(f,x) - f(x)\right| \leq 2 \left(1 + x^2\right) \left[ A_1 \left(1 + x^2\right)^2 + \sqrt{D_1} \frac{1}{\delta} \left(1 + x^2\right)^2 \right] \Omega_2(f,\delta).
\]

Now, by taking \( \delta = \frac{1}{\sqrt{D_1}} \), \( A_1 = 2 \left(1 + A_1 + A_2\right) \), and we get the desired result.

3. Conclusion

In this paper, by using the notion of \((p,q)\)-calculus and weighted modulus of continuity we study weighted approximation properties of new \((p,q)\)-analogue of the Balázs-Szabados operators. We give the rate of convergence for these operators and we give a Korovkin type theorem for weighted approximation.

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References


Theory, 5, 92-164.

