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Rotation Minimizing Frame and Rectifying Curves in E_1^n

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Abstract

In this paper, some applications of a Rotation minimizing frame (RMF) are studied in E_1^4 and in E_1^n for timelike, spacelike curves. Firstly, in E_1^4, a Rotation minimizing frame (RMF) is obtained on the timelike and spacelike direction curves $\int N(s)\, ds$. The features of this Rotation minimizing frame are expressed. Secondly, it is determined when the timelike and spacelike curves can be rectifying curves. In addition, it has been investigated the conditions under which timelike and spacelike curves can be sphere calcurves. Then, a new characterization of rectifying curves is given, similar to the characterization of spherical curves. Finally, this Rotation minimizing frame is generalized in E_1^n for timelike, spacelike curves. In E_1^n, the conditions being a spherical curve and arectifying curve are given thank to this frame for timelike and spacelike curves. Also, a relationship between the spherical curve and the rectifying curve is stated. It is shown that the coefficients used in obtaining rectifying curves are constant numbers.

Keywords

Rectifying curve, spherical curve, rotation minimizing frame (RMF)

1. Introduction

Rotation minimizing frame (RMF) is introduced by Bishop as an alternative to the Frenet moving frame along a curve γ in n-dimensional Euclidean space E^n. The Frenet frame is an orthonormal frame which can be defined for curves in E^n. As a result, Frenet frame and Rotation minimizing frame (RMF) are orthonormal frames. An RMF along a curve $\beta = \beta(s)$ in E^n is defined by the tangent vector and (n-1) normal vector fields N_i so that $N'_i(s)$ are proportional to $\beta'(s)$. Such a normal vector field along a curve is called a Rotation minimizing vector field. Any orthonormal basis $\{\beta'(s_0), N_1(s_0), \ldots, N_{n-1}(s_0)\}$ at a point $\beta(s_0)$ expresses a unique RMF along the curve γ. Hence, such an RMF is uniquely designated modula a rotation in E^{n-1} [1, 2, 3, 4, 5].

Recently, RM frames is widely used in computer graphics, including sweep or blending surface modeling, motion design and control in computer animation and robotics, etc. This issue has begun to attract attention among researchers. Let us briefly expressed some of them. A novel simple and efficient method for accurate and stable computation of an RMF of a curve in 3D is expressed. RM frames of space curves are used for sweep surface modelling [2]. It is proved that a normal vector field of a curve is a Rotation minimizing (RM) if and only if it is parallel with regard the normal connection. Therefore, all the results of RM vectors and frames are generalized to curves immersed in Riemannian manifolds. RM vector fields along a curve immersed into a Riemannian manifold are studied when the ambient manifold is the Euclidean 3-space, the Hyperbolic 3-space and a Kähler manifold [3]. But, we have seen that an RMF has not been studied much in the papers. So, we decided to review this frame in the Minkowski spaces.

In this studied, some applications of a Rotation minimizing frame (RMF) are studied in E_1^4 and in E_1^n for timelike and spacelike curves. In E_1^4, a Rotation minimizing frame (RMF) is obtained on $\int N(s)\,ds$ direction timelike and spacelike curves. The properties of this Rotation minimizing frame (RMF) are expressed. The condition being a spherical curve is given with the help of this frame for timelike and spacelike curves. This Rotation minimizing frame (RMF) is generalized in E_1^n. Secondly, this Rotation minimizing frame (RMF) is applied to timelike and spacelike rectifying curves in E_1^4 and E_1^n. Although the coefficients of timelike and spacelike rectifying curves are functions in another papers [6, 7], this coefficients of timelike and spacelike rectifying curve are constants in our paper. Finally, relationship between spherical curve and rectifying curve is given.

2. Preliminaries

The Minkowski 4-space E_1^4 is the real vector space R^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{2.1}$$

where (x_1, x_2, x_3, x_4) is rectangular coordinate system of E_1^4. Since g is an indefinite metric, recall that a vector $v \neq 0$ in E_1^4 can be a spacelike, a timelike or null(lightlike), if respectively holds g(v, v) > 0, g(v,v) < 0 or g(v, v) = 0. In particular, the vector v = 0 is a spacelike. The norm (length) of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$ and two vectors v and ω are said to be orthonormal when g(v, ω) = 0. Also, an arbitrary curve $\alpha = \alpha(s)$ can locally be a spacelike, timelike or null (lightlike), if all of its velocity $\alpha'(s)$ are respectively spacelike, timelike or null [8].

Let $\{T, N, V_3, V_4\}$ be the Frenet frame along a unit-speed timelike curve α in E_1^4. Then, the Frenet-Serret formulas are given as follows:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ V'_3(s) \\ V'_4(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ V_3(s) \\ V_4(s) \end{bmatrix}, \tag{2.2}$$

where $\langle T, T\rangle = -1$, $\langle N, N\rangle = \langle V_3, V_3\rangle = \langle V_4, V_4\rangle = 1$, $\langle T, N\rangle = \langle T, V_3\rangle = \langle T, V_4\rangle = \langle V_3, V_4\rangle = 0$. Here k_1, k_2 and k_3 are the principal curvatures of α. Such that $k_1(s) = \langle T'(s), N\rangle$, $k_2(s) = -\langle V_3'(s), N\rangle$ and $k_3(s) = -\langle V_4'(s), V_3\rangle$, see [9].

Let $\{T, N, V_3, V_4\}$ be the Frenet frame along a unit-speed spacelike curve α in E_1^4. Also, V_4 is the unique-timelike unit vector field. Then, the Frenet-Serret formulas are given as follows:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ V'_3(s) \\ V'_4(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ V_3(s) \\ V_4(s) \end{bmatrix}, \tag{2.3}$$

where $\langle T, T\rangle = \langle N, N\rangle = \langle V_3, V_3\rangle = 1$, $\langle V_4, V_4\rangle = -1$ $\langle T, N\rangle = \langle T, V_3\rangle = \langle T, V_4\rangle = \langle V_3, V_4\rangle = 0$. Here k_1, k_2 and k_3 are the principal curvatures of α. Such that $k_1(s) = \langle T'(s), N\rangle$, $k_2(s) = -\langle V_3'(s), N\rangle$ and $k_3(s) = \langle V_4'(s), V_3\rangle$, see [9].

Let $\{T, N, V_3, V_4\}$ be the Frenet frame along a unit-speed spacelike curve α in E_1^4. Also, N is the unique-timelike unit vector field. Then, the Frenet-Serret formulas are given as follows:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ V'_3(s) \\ V'_4(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ V_3(s) \\ V_4(s) \end{bmatrix}, \tag{2.4}$$

where $\langle T, T\rangle = \langle V_3, V_3\rangle = \langle V_4, V_4\rangle = 1$, $\langle N, N\rangle = -1$, $\langle T, N\rangle = \langle T, V_3\rangle = \langle T, V_4\rangle = \langle V_3, V_4\rangle = 0$. Here k_1, k_2 and k_3 are the principal curvatures of α. Such that $k_1(s) = -\langle T'(s), N\rangle$, $k_2(s) = -\langle V_3'(s), N\rangle$ and $k_3(s) = -\langle V_4'(s), V_3\rangle$, see [9].

Definition 2.1. A normal vector field V = V(s) over a curve $\gamma = \gamma(s)$ in E^n is said to be relatively parallel or Rotation minimizing (RM) if the derivative V'(s) is proportional to $\gamma'(s)$, see [2, 3].

Remark 2.1. Let a normal vector field V = V(s) over a curve $\gamma = \gamma(s)$ in E^3. Then, the following statements can be written.

- Let $\gamma = \gamma(s)$ is a curve with Rotation minimizing normal vector field V = V(s) in E^3. In this case the ruled surface $f(s, \lambda) = \gamma(s) + \lambda V(s)$ is developable, because $det[\gamma'(s), V(s), V'(s)] = 0$.
- If V is a RM vector field, then $\|V\|$ is a constant. Let T denote the tangent vector to γ, then

V'=λT $\Rightarrow$ $V' \perp V$ $\Rightarrow \frac{d}{dt}(V.V) = 0$, see [2, 3]

Definition 2.2. Let $\gamma = \gamma(s)$ be a curve in E^n. An RM frame, a parallel frame, a natural frame, a Bishop frame or an adapted frame is a moving orthonormal frame {T(s), N_i(s)}, i = 1,2,...,n-1 along γ, where T(s) is the tangent vector to γ at the point $\gamma(s)$ and N_i(s) = { N_1(s), N_2(s),..., N_{n-1}(s)} are RM vector fields, see [2, 3].

If $\gamma = \gamma(s)$, γ'(s) = T and u(s) is an RM vector field, then $T \wedge u$ is an RM vector field. Thus, {T, u, $T \wedge u$} is an RM frame. This type frame is defined in E^4, see [10]. Moreover, in [10], an RM frame is obtained using Euler angles in E^4. Now, we will define RMF using a new method along spacelike curve and timelike curve.

3. Rotation Minimizing Frame in $\mathbf{E_1^4}$

In this section, we will investigate a Rotation minimizing frame (RMF) for fourth case. In E_1^4, as the curve changes, we'll show how to change of obtain an RMF.

Case 1. Let $\alpha = \alpha(s)$ be a unit-speed regular timelike curve in E_1^4. Equation (2.2) is the Frenet frame of this curve. If V_3 and V_4 are rotated in the plane Sp$\{V_3, V_4\}$, then

$$\begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} V_3(s) \\ V_4(s) \end{bmatrix},$$

where $\theta(s) = -\int k_3(s)\,ds$. Calculating the derivatives N_1andN_2, we get $N'_1 = -k_2\cos\theta(s)\,N$ and $N'_2 = k_2\sin\theta(s)\,N$. Also, if $\bar{k}_1 = k_2\cos\theta(s)$ and $\bar{k}_2 = k_2\sin\theta(s)$, then $N'_1 = -\bar{k}_1 N$ and $N'_2 = \bar{k}_2 N$. In addition, one can be written $\bar{k}_1{}^2 + \bar{k}_2{}^2 = k_2{}^2$ and $\theta(s) = \arctan\left(\frac{\bar{k}_2}{\bar{k}_1}\right)$.

The formulae of the new frame can be given as

$$\begin{bmatrix} N'(s) \\ N'_1(s) \\ N'_2(s) \\ T'(s) \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_1 & -\bar{k}_2 & k_1 \\ -\bar{k}_1 & 0 & 0 & 0 \\ \bar{k}_2 & 0 & 0 & 0 \\ k_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ N_1(s) \\ N_2(s) \\ T(s) \end{bmatrix}. \tag{3.1}$$

Since the derivatives of T, N_1 and N_2 are in the same direction with α' (*i.e.* in the tangential direction of α), the frame {T, N, N_1, N_2} is called as **"Rotation Minimizing Frame (RMF)"** on the direction curve $\int$N(s) ds. Here $\bar{k}_1$, $\bar{k}_2$ and k_1 are the Rotation minimizing curvatures of $\int$N(s) ds.

We can generalize the frame given by Equation (3.1) as

$$\begin{bmatrix} N'(s) \\ N'_1(s) \\ N'_2(s) \\ \dots \\ N'_{n-2}(s) \\ T'(s) \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_1 & -\bar{k}_2 & \dots & -\bar{k}_{n-2} & k_1 \\ -\bar{k}_1 & 0 & 0 & \dots & 0 & 0 \\ \bar{k}_2 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \bar{k}_{n-2} & 0 & 0 & \dots & 0 & 0 \\ k_1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ N_1(s) \\ N_2(s) \\ \dots \\ N_{n-2}(s) \\ T(s) \end{bmatrix}. \tag{3.2}$$

This generalized frame is an RMF on the direction curve $\int$N(s) ds.

Case 2. Let $\alpha = \alpha(s)$ be a unit-speed regular spacelike curve in E_1^4. Also, V_4 is the unique-timelike unit vector field. Equation (2.3) is the Frenet frame of this curve. If V_3 and V_4 are rotated in the plane Sp$\{V_3, V_4\}$, then

$$\begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} \cosh\theta(s) & \sinh\theta(s) \\ \sinh\theta(s) & \cosh\theta(s) \end{bmatrix} \begin{bmatrix} V_3(s) \\ V_4(s) \end{bmatrix},$$

where $\theta(s) = -\int k_3(s)\,ds$. Calculating the derivatives $N_1$and N_2, we get $N'_1 = -k_2\cosh\theta(s)\,N$and $N'_2 = -k_2\sinh\theta(s)\,N$. Moreover, if $\bar{k}_1 = k_2\cosh\theta(s)$ and $\bar{k}_2 = k_2\sinh\theta(s)$, then $N'_1 = -\bar{k}_1 N$ and $N'_2 = -\bar{k}_2 N$. In addition, one can be written $\bar{k}_1{}^2 - \bar{k}_2{}^2 = k_2{}^2$ and $\theta(s) = \operatorname{arctanh}\left(\frac{\bar{k}_2}{\bar{k}_1}\right)$.

The formulae of the new frame can be given as

$$\begin{bmatrix} N'(s) \\ N'_1(s) \\ N'_2(s) \\ T'(s) \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_1 & -\bar{k}_2 & -k_1 \\ -\bar{k}_1 & 0 & 0 & 0 \\ -\bar{k}_2 & 0 & 0 & 0 \\ k_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ N_1(s) \\ N_2(s) \\ T(s) \end{bmatrix}, \tag{3.3}$$

Since the derivatives of T, N_1 and N_2 are in the same direction with α' (*i.e.* in the tangential direction of α), the frame {T, N, N_1, N_2} is called as **"Rotation Minimizing Frame (RMF)"** on the direction curve $\int$N(s) ds. Here $\bar{k}_1$, $\bar{k}_2$ and k_1 are the Rotation minimizing curvatures of $\int$N(s) ds.

We can generalize the frame given by Equation (3.3) as

$$\begin{bmatrix} N'(s) \\ N'_1(s) \\ N'_2(s) \\ \dots \\ N'_{n-2}(s) \\ T'(s) \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_1 & -\bar{k}_2 & \dots & -\bar{k}_{n-2} & -k_1 \\ -\bar{k}_1 & 0 & 0 & \dots & 0 & 0 \\ -\bar{k}_2 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\bar{k}_{n-2} & 0 & 0 & \dots & 0 & 0 \\ k_1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ N_1(s) \\ N_2(s) \\ \dots \\ N_{n-2}(s) \\ T(s) \end{bmatrix}. \tag{3.4}$$

This generalized frame is an RMF on the direction curve $\int$ N(s) ds.

Case 3. Let $\alpha = \alpha(s)$ be a unit-speed regular spacelike curve in E_1^4. Also, N is the unique-timelike unit vector field. Equation (2.4) is the Frenet frame of this curve. If V_3 and V_4 are rotated in the plane Sp$\{V_3, V_4\}$, then

$$\begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} V_3(s) \\ V_4(s) \end{bmatrix},$$

where $\theta(s) = -\int k_3(s)ds$. Calculating the derivatives $N_1$and N_2, we get $N'_1 = k_2\cos\theta(s)\, N$and $N'_2 = -k_2\sin\theta(s)\, N$. Nonetheless, if $\bar{k}_1 = k_2\cos\theta(s)$ and $\bar{k}_2 = k_2\sin\theta(s)$, then $N'_1 = \bar{k}_1 N$ and $N'_2 = -\bar{k}_2 N$. In addition, one can be written $\bar{k}_1{}^2 + \bar{k}_2{}^2 = k_2{}^2$ and $\theta(s) = \arctan\left(\frac{\bar{k}_2}{\bar{k}_1}\right)$.

The formulae of the new frame can be given as

$$\begin{bmatrix} N'(s) \\ N'_1(s) \\ N'_2(s) \\ T'(s) \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_1 & -\bar{k}_2 & k_1 \\ \bar{k}_1 & 0 & 0 & 0 \\ -\bar{k}_2 & 0 & 0 & 0 \\ k_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ N_1(s) \\ N_2(s) \\ T(s) \end{bmatrix}. \tag{3.5}$$

Since the derivatives of T, N_1 and N_2 are in the same direction with α' (*i.e.* in the tangential direction of α), the frame {T, N, N_1, N_2} is called as **"Rotation Minimizing Frame (RMF)"** on the direction curve $\int$ N(s) ds. Here $\bar{k}_1$, $\bar{k}_2$ and k_1 are the Rotation minimizing curvatures of $\int$ N(s) ds.

We can generalize the frame given by Equation (3.5) as

$$\begin{bmatrix} N'(s) \\ N'_1(s) \\ N'_2(s) \\ \dots \\ N'_{n-2}(s) \\ T'(s) \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_1 & -\bar{k}_2 & \dots & -\bar{k}_{n-2} & k_1 \\ \bar{k}_1 & 0 & 0 & \dots & 0 & 0 \\ -\bar{k}_2 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\bar{k}_{n-2} & 0 & 0 & \dots & 0 & 0 \\ k_1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ N_1(s) \\ N_2(s) \\ \dots \\ N_{n-2}(s) \\ T(s) \end{bmatrix}. \tag{3.6}$$

This generalized frame is an RMF on the direction curve $\int$ N(s) ds. Thus, we are given the below theorems.

Theorem 3.1. Let $\alpha: I \to E_1^4$ be a spacelike curve (or a timelike curve) and $\{T, N, V_3, V_4\}$ be the Frenet frame of α. Then, the frame {T, N, N_1, N_2} is an RMF on the direction curve $\gamma = \int$ N(s) ds.

Proof. The proof is obvious from Equation (3.1), Equation (3.3) and Equation (3.5).

Theorem 3.2. Direction curve $\gamma = \int$ N(s) ds is a timelike spherical curve (or a spacelike spherical curve) if and only if $\lambda_1\bar{k}_1 + \lambda_2\bar{k}_2 + \lambda_3 k_1 + 1 = 0$. Here $\bar{k}_1$, $\bar{k}_2$ and k_1 are the Rotation minimizing curvatures and λ_1, λ_2, λ_3 are constants.

Proof. Let $\gamma = \int$ N(s) ds be a timelike spherical curve (or a spacelike spherical curve). Then, $\gamma(s) = \lambda_1 N_1 + \lambda_2 N_2 + \lambda_3$T. If the derivative of both sides are taken and if the necessary calculations are done, we can easily obtain $\lambda_1\bar{k}_1 + \lambda_2\bar{k}_2 + \lambda_3 k_1 + 1 = 0$. Here λ_1, λ_2 and λ_3 are constants.

Theorem 3.3. Direction curve $\gamma = \int$ N(s) ds is a timelike spherical curve (or a spacelike spherical curve) if and only if $\lambda_1\bar{k}_1 + \lambda_2\bar{k}_2 + \dots + \lambda_{n-2}\bar{k}_{n-2} + \lambda_{n-1}k_1 + 1 = 0$. Here $\bar{k}_1, \bar{k}_2, \dots, \bar{k}_{n-2}, k_1$ are the Rotation minimizing curvatures and $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ are constants.

Proof. The proof can be given like the proof of Theorem (3.2).

4. Rectifying Curves and Rotation Minimizing Frame on E_1^4

In the Minkowski 4-space E_1^4, rectifying curve is defined as a curve whose position vector always lies in orthogonal complement $N^\perp$ of its principal normal vector field N. Consequently,

$$N^\perp = \{W \in E_1^4 \mid \langle W, N\rangle = 0\},$$

where $\langle , \rangle$ denotes the standard pseudo scalar product in E_1^4. Hence $N^\perp$ is a 3-dimensional subspace of E_1^4, spanned by the tangent, the first binormal and the second binormal vector fields T, B_1 and B_2, respectively. Hence, the position vector with respect to some chosen origin, of a rectifying spacelike curve α in E_1^4, satisfies the equation,

$$\alpha(s) = \lambda(s)\mathrm{T}(s) + \mu(s)B_1(s) + \eta(s)B_2(s),$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\eta(s)$ in arclength function s. Then, rectifying spacelike curves in terms of their curvature functions $k_1(\mathrm{s})$, $k_2(\mathrm{s})$, $k_3(\mathrm{s})$ and the necessary and the sufficient conditions for an arbitrary curve in E_1^4 to be a rectifying is given [7].

Theorem 4.1. Let α be a unit speed spacelike curve in E_1^4, with non-zero curvatures $k_1(\mathrm{s})$, $k_2(\mathrm{s})$ and $k_3(\mathrm{s})$. Then α is congruent to a rectifying spacelike curve if and only if

$$\frac{(s+c)k_1(\mathrm{s})}{k_2(\mathrm{s})} = \epsilon(A\cosh\int_0^s k_3(\mathrm{s})\mathrm{ds} + \mathrm{B}\sinh\int_0^s k_3(\mathrm{s})\mathrm{ds}). \tag{4.1}$$

In particular, assume that all curvature functions $k_1(\mathrm{s})$, $k_2(\mathrm{s})$ and $k_3(\mathrm{s})$ of rectifying spacelike curve α in E_1^4, are constants and different from zero [11].

Theorem 4.2. There are no rectifying spacelike curves lying in E_1^4, with non-zero constant curvatures $k_1(\mathrm{s})$, $k_2(\mathrm{s})$ and $k_3(\mathrm{s})$ [11].

Now, we will give a new characterization of rectifying curves using the RMF in E_1^4.

Theorem 4.3. Let α be a timelike curve in E_1^4 and let $\{\mathrm{T}, \mathrm{N}, N_1, N_2\}$ be an RMF on the direction curve $\gamma(s) = \int \mathrm{N}(\mathrm{s})\,\mathrm{ds}$. Then, α is a timelike rectifying curve in E_1^4 if and only if $A\,\bar{k}_1 - B\bar{k}_2 = (s+b)k_1$, where $A,\ B\ \in R$, $\bar{k}_1$, $\bar{k}_2$ and k_1 are the Rotation minimizing curvatures.

Proof. $(\Longrightarrow)$ Let $\alpha = \alpha(s)$ be a timelike rectifying curve in E_1^4. Then,

$$\alpha(s) = \lambda_1\mathrm{T} + \lambda_2 N_1 + \lambda_3 N_2.$$

Taken derivatives of both sides, we get

$$T = \alpha' = \lambda'_1 T + \lambda_1 T' + \lambda'_2 N_1 + \lambda_2 N'_1 + \lambda'_3 N_2 + \lambda_3 N'_2.$$

Substituting $T' = k_1 N$, $N'_1 = -\bar{k}_1\mathrm{N}$ and $N'_2 = \bar{k}_2\mathrm{N}$, we obtain

$$\lambda'_1 = 1, \lambda'_2 = 0, \lambda'_3 = 0,$$

and

$$\lambda_1 k_1 - \lambda_2\bar{k}_1 + \lambda_3\bar{k}_2 = 0.$$

Consequently, since $\lambda'_1 = 1$, $\lambda'_2 = 0$ and $\lambda'_3 = 0$, it is obtained that $\lambda_1 = s+b$, $\lambda_2 = A$ and $\lambda_3 = B$, $A,\ B\ \in R$. Thus, $A\,\bar{k}_1 - B\bar{k}_2 = (s+b)k_1$.

$(\Longleftarrow)$ Let $A\,\bar{k}_1 - B\bar{k}_2 = (s+b)k_1$, for $A,\ B\ \in R$ and let $\bar{k}_1$, $\bar{k}_2$ and k_1 be the Rotation minimizing curvatures. Since

$$\frac{d}{ds}(\alpha(s) - (s+b)T - AN_1 - BN_2) = 0,$$

$\alpha(s) = (s+b)T + AN_1 + BN_2 + \vec{C}$ is a timelike rectifying curve in E_1^4. Hence, the proof is completed.

Theorem 4.4. Let α be a spacelike curve (V_4 is the unique-timelike unit vector field) in E_1^4 and let $\{\mathrm{T}, \mathrm{N}, N_1, N_2\}$ be an RMF on the direction curve $\gamma(s) = \int \mathrm{N}(\mathrm{s})\,\mathrm{ds}$. Then, α is a spacelike rectifying curve in E_1^4 if and only if $A\,\bar{k}_1 + B\bar{k}_2 = (s+b)k_1$, where $A,\ B\ \in R$, $\bar{k}_1$, $\bar{k}_2$ and k_1 are the Rotation minimizing curvatures.

Proof. $(\Longrightarrow)$ Let $\alpha = \alpha$ (s) be a spacelike rectifying curve in E_1^4. Then,

$$\alpha(s) = \lambda_1\mathrm{T} + \lambda_2 N_1 + \lambda_3 N_2.$$

Taken derivatives of both sides, we get

$$T = \alpha' = \lambda'_1 T + \lambda_1 T' + \lambda'_2 N_1 + \lambda_2 N'_1 + \lambda'_3 N_2 + \lambda_3 N'_2.$$

Substituting $T' = k_1 N$, $N'_1 = -\bar{k}_1\mathrm{N}$ and $N'_2 = -\bar{k}_2\mathrm{N}$, we obtain

$$\lambda'_1 = 1, \qquad \lambda'_2 = 0, \qquad \lambda'_3 = 0,$$

and

$$\lambda_1 k_1 - \lambda_2\bar{k}_1 - \lambda_3\bar{k}_2 = 0.$$

Consequently, since $\lambda'_1 = 1$, $\lambda'_2 = 0$ and $\lambda'_3 = 0$, it is obtained that $\lambda_1 = s+b$, $\lambda_2 = A$ and $\lambda_3 = B$, $A,\ B\ \in R$. Thus, $A\,\bar{k}_1 + B\bar{k}_2 = (s+b)k_1$.

(⟸) Let $A\,\bar{k}_1 + B\bar{k}_2 = (s+b)k_1$, for $A,\ B\ \in\ R$ and let $\bar{k}_1,\ \bar{k}_2$ and k_1 be the Rotation minimizing curvatures. Since

$$\frac{d}{ds}(\alpha(s) - (s+b)T - AN_1 - BN_2) = 0,$$

$\alpha(s) = (s+b)T + AN_1 + BN_2 + \vec{C}$ is a spacelike rectifying curve in E_1^4. Hence, the proof is completed.

Theorem 4.5. Let α be a spacelike curve (N is the unique-timelike unit vector field) in E_1^4 and let $\{T,\ N,\ N_1,\ N_2\}$ be an RMF on the direction curve $\gamma(s) = \int N(s)\,ds$. Then, α is a spacelike rectifying curve in E_1^4 if and only if $B\,\bar{k}_2 - A\bar{k}_1 = (s+b)k_1$, where $A,\ B\ \in\ R,\ \bar{k}_1,\ \bar{k}_2$ and k_1 are the Rotation minimizing curvatures.

Proof. (⟹) Let $\alpha = \alpha$ (s) be a spacelike rectifying curve in E_1^4. Then,

$$\alpha(s) = \lambda_1 T + \lambda_2 N_1 + \lambda_3 N_2.$$

Taken derivatives of both sides, we get

$$T = \alpha' = \lambda'_1 T + \lambda_1 T' + \lambda'_2 N_1 + \lambda_2 N'_1 + \lambda'_3 N_2 + \lambda_3 N'_2.$$

Substituting $T' = k_1 N,\ N'_1 = \bar{k}_1 N$ and $N'_2 = -\bar{k}_2 N$, we obtain

$$\lambda'_1 = 1, \lambda'_2 = 0, \lambda'_3 = 0,$$

and

$$\lambda_1 k_1 + \lambda_2 \bar{k}_1 - \lambda_3 \bar{k}_2 = 0.$$

Consequently, since $\lambda'_1 = 1,\ \lambda'_2 = 0$ and $\lambda'_3 = 0$, it is obtained that $\lambda_1 = s+b,\ \lambda_2 = A$ and $\lambda_3 = B,\ A,\ B\ \in\ R$. Thus, $B\,\bar{k}_2 - A\bar{k}_1 = (s+b)k_1$.

(⟸) Let $B\,\bar{k}_2 - A\bar{k}_1 = (s+b)k_1$, for $A,\ B\ \in\ R$ and let $\bar{k}_1,\ \bar{k}_2$ and k_1 be the Rotation minimizing curvatures. Since

$$\frac{d}{ds}(\alpha(s) - (s+b)T - AN_1 - BN_2) = 0,$$

$\alpha(s) = (s+b)T + AN_1 + BN_2 + \vec{C}$ is a spacelike rectifying curve in E_1^4. Hence, the proof is completed.

Theorem 4.6. A C^4-curve $x = x(s),\ s \in [0, L]$, parametrized by arc length s with curvature k_1 and torsion k_2 is a spherical curve if and only if

$$(A\cos\textstyle\int_0^s k_2 ds + B\sin\int_0^s k_2 ds) = k_1^{-1}(s), \tag{4.2}$$

where A, B are constants, see [12].

Theorem 4.7. A unit speed timelike curve $x = x(s)$ with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$ lies on a Lorentzian sphere in E_1^3 if and only if there are constants $A,\ B\ \in\ R$ such that the equation

$$(A\cos\textstyle\int_0^s \tau\,ds + B\sin\int_0^s \tau\,ds) = \kappa^{-1}(s), \tag{4.3}$$

holds for each $s \in I \subset R$ [13].

Theorem 4.8. A unit speed spacelike curve $x = x(s)$ with the timelike principal normal N, lies on a Lorentzian sphere if and only if there are constants $A,\ B\ \in\ R$ such that

$$(A\sinh\textstyle\int_0^s \tau\,ds - B\cosh\int_0^s \tau\,ds) = \kappa^{-1}(s), \tag{4.4}$$

holds for each $s \in I \subset R$ [14].

Now, we will give the characterization of rectifying curves similar to the characterization of spherical curves.

Theorem 4.9. A C^4- timelike curve $\alpha = \alpha(s)$ in $E_1^4,\ s \in [0, L]$, parametrized by its arc length s with curvatures k_1, k_2 and k_3 is a timelike rectifying curve (according to **Case 1.**) if and only if

$$(A\cos\textstyle\int_0^s k_3 ds - B\sin\int_0^s k_3 ds) = \frac{(s+b)k_1}{k_2}, \tag{4.5}$$

where A, B are constants.

Proof. Let $\alpha = \alpha(s)$ be a timelike rectifying curve having the curvatures $k_1,\ k_2$ and k_3 in E_1^4. Then, according to Theorem 4.3, it is obtained that $A\,\bar{k}_1 - B\bar{k}_2 = (s+b)k_1$, where $A,\ B\ \in\ R$. Also, for timelike curve, since $\bar{k}_1 = k_2\cos\theta(s),\ \bar{k}_2 = k_2\sin\theta(s)$ and $\theta(s) = \int k_3(s)\,ds$, we get

$$(A\cos\int_0^s k_3 ds - B\sin\int_0^s k_3 ds) = \frac{(s+b)k_1}{k_2},$$

where A, B are constants. Conversely, let

$$(A\cos\int_0^s k_3\mathrm{ds} - \mathrm{B}\sin\int_0^s k_3\mathrm{ds}) = \frac{(s+b)k_1}{k_2},$$

where A, B are constants. If $\int_0^s k_3(s)\ ds = \theta$, then

$$(A\cos\theta - \mathrm{B}\sin\theta) = \frac{(s+b)k_1}{k_2}.$$

If the necessary calculations are done, it is found

$$(A\,k_2\cos\theta - \mathrm{B}\,k_2\sin\theta) = (s+b)k_1.$$

If $\bar{k}_1 = k_2\cos\theta(s)$ and $\bar{k}_2 = k_2\sin\theta(s)$, then Theorem 4.3 is obtained. That is to say, $A\,\bar{k}_1 - B\bar{k}_2 = (s+b)k_1$, where $A,\ B\ \in\ R$. Also, $\bar{k}_1 = k_2\cos\theta(s)$, $\bar{k}_2 = k_2\sin\theta(s)$ and $\theta(s) = \int k_3(s)\,ds$. This theorem shows that the curve is a timelike rectifying curve.

Result 4.1. If $k_3 = 0$, then $\theta = 0$. Since $\bar{k}_1 = k_2\cos\theta(s)$ and $\bar{k}_2 = k_2\sin\theta(s)$, it is $\bar{k}_1 = k_2$ and $\bar{k}_2 = 0$. Thus, $A\,k_2 = (s+b)k_1$. In addition, since $A\,k_2 = (s+b)k_1$, it is $\frac{k_2}{k_1} = \frac{1}{A}(s+b)$.

Then, in Theorem 4.9, the given curve is a timelike rectifying curve in E_1^3, see [15].

Theorem 4.10. A C^4- spacelike curve $\alpha = \alpha(s)$ in E_1^4, $s \in [0,\mathrm{L}]$, parametrized by its arc length s with curvatures k_1, k_2 and k_3 is a spacelike rectifying curve (according to **Case 2.**) if and only if

$$(A\cosh\textstyle\int_0^s k_3\mathrm{ds} - \mathrm{B}\sinh\int_0^s k_3\mathrm{ds}) = \frac{(s+b)k_1}{k_2}, \tag{4.6}$$

where A, B are constants.

Proof. Let $\alpha = \alpha(s)$ be a spacelike rectifying curve having the curvatures k_1, k_2 and k_3 in E_1^4. Then, according to Theorem 4.4, it is obtained that $A\,\bar{k}_1 + B\bar{k}_2 = (s+b)k_1$, where $A,\ B\ \in\ R$. Also, for spacelike curve, since $\bar{k}_1 = k_2\cosh\theta(s)$, $\bar{k}_2 = k_2\sinh\theta(s)$ and $\theta(s) = -\int k_3(s)\,ds$, we get

$$(A\cosh\int_0^s k_3\mathrm{ds} - \mathrm{B}\sinh\int_0^s k_3\mathrm{ds}) = \frac{(s+b)k_1}{k_2},$$

where A, B are constants. Conversely, let

$$(A\cosh\int_0^s k_3\mathrm{ds} - \mathrm{B}\sinh\int_0^s k_3\mathrm{ds}) = \frac{(s+b)k_1}{k_2},$$

where A, B are constants. If (- $\int_0^s k_3\,(s)\,ds) = \theta$, then

$$(A\cosh\theta + \mathrm{B}\sinh\theta) = \frac{(s+b)k_1}{k_2}.$$

If the necessary calculations are done, it is found

$$(A\,k_2\cosh\theta + \mathrm{B}\,k_2\sinh\theta) = (s+b)k_1.$$

If $\bar{k}_1 = k_2\cosh\theta(s)$ and $\bar{k}_2 = k_2\sinh\theta(s)$, then Theorem 4.4 is obtained. That is to say, $A\,\bar{k}_1 + B\bar{k}_2 = (s+b)k_1$, where $A,\ B\ \in\ R$. Also, $\bar{k}_1 = k_2\cosh\theta(s)$, $\bar{k}_2 = k_2\sinh\theta(s)$ and $\theta(s) = -\int k_3(s)\,ds$. This theorem shows that the curve is a spacelike rectifying curve.

Result 4.2. If $k_3 = 0$, then $\theta = 0$. Since $\bar{k}_1 = k_2\cosh\theta(s)$ and $\bar{k}_2 = k_2\sinh\theta(s)$, it is $\bar{k}_1 = k_2$ and $\bar{k}_2 = 0$. Thus, $A\,k_2 = (s+b)k_1$. In addition, since $A\,k_2 = (s+b)k_1$, it is $\frac{k_2}{k_1} = \frac{1}{A}(s+b)$.

Then, in Theorem 4.10, the given curve is a spacelike rectifying curve in E_1^3, see [15].

Theorem 4.11. A C^4- spacelike curve $\alpha = \alpha(s)$ in E_1^4, $s \in [0,\mathrm{L}]$, parametrized by its arc length s with curvatures k_1, k_2 and k_3 is a spacelike rectifying curve (according to **Case 3.**) if and only if

$$(B\sin\textstyle\int_0^s(-k_3)\mathrm{ds} - \mathrm{A}\cos\int_0^s(-k_3)\mathrm{ds}) = \frac{(s+b)k_1}{k_2}, \tag{4.7}$$

where A, B are constants.

Proof. Let $\alpha = \alpha(s)$ be a spacelike rectifying curve having the curvatures k_1, k_2 and k_3 in E_1^4. Then, according to Theorem 4.5, it is obtained that $B\,\bar{k}_2 - A\bar{k}_1 = (s+b)k_1$, where $A,\ B\ \in\ R$. Also, for spacelike curve, since $\bar{k}_1 = k_2\cos\theta(s)$, $\bar{k}_2 = k_2\sin\theta(s)$ and $\theta(s) = -\int k_3(s)\,ds$, we get

$$(B\sin\int_0^s(-k_3)\text{ds}-\text{A}\cos\int_0^s(-k_3)\text{ds})=\frac{(s+b)k_1}{k_2},$$

where A, B are constants. Conversely, let

$$(B\sin\int_0^s(-k_3)\text{ds}-\text{A}\cos\int_0^s(-k_3)\text{ds})=\frac{(s+b)k_1}{k_2},$$

where A, B are constants. If (- $\int_0^s k_3(s)\ ds)=\theta$, then

$$(B\sin\theta-\text{A}\cos\theta)=\frac{(s+b)k_1}{k_2}.$$

If the necessary calculations are done, it is found

$$(B\,k_2\sin\theta-\text{A}\,k_2\cos\theta)=(s+b)k_1.$$

If $\bar{k}_1=k_2\cos\theta(s)$ and $\bar{k}_2=k_2\sin\theta(s)$, then Theorem 4.5 is obtained. That is to say, $B\,\bar{k}_2-A\bar{k}_1=(s+b)k_1$, where $A,\ B\ \in\ R$. Also, $\bar{k}_1=k_2\cos\theta(s)$, $\bar{k}_2=k_2\sin\theta(s)$ and $\theta(s)=-\int k_3(s)\,ds$. This theorem shows that the curve is a spacelike rectifying curve.

Result 4.3. If $k_3=0$, then $\theta=0$. Since $\bar{k}_1=k_2\cos\theta(s)$ and $\bar{k}_2=k_2\sin\theta(s)$, it is $\bar{k}_1=k_2$ and $\bar{k}_2=0$. Thus, $-A\,k_2=(s+b)k_1$. In addition, since $-A\,k_2=(s+b)k_1$, it is $\frac{k_2}{k_1}=-\frac{1}{A}(s+b)$.

Then, in Theorem 4.11, the given curve is a spacelike rectifying curve in E_1^3, see [15].

Now, we can give the following theorems.

Theorem 4.12. Let $\alpha=\alpha(s)$ be a timelike curve having the curvatures k_1, k_2 and k_3 in E_1^4. And let $\beta=\beta(s)$ be a timelike curve in E_1^3 having $\tau=-k_3$, $\kappa=\frac{k_2}{k_1(s+b)}$. Hence, $\alpha=\alpha(s)$ is a timelike rectifying curve in E_1^4 (according to **Case 1.**) if and only if $\beta=\beta(s)$ is a timelike spherical curve in E_1^3.

Proof. Let $\alpha=\alpha(s)$ be a timelike rectifying curve in E_1^4. In this case, according to Theorem 4.9,

$$(A\cos\int_0^s k_3\text{ds}-\text{B}\sin\int_0^s k_3\text{ds})=\frac{(s+b)k_1}{k_2},$$

If $\tau=-k_3$, $\kappa=\frac{k_2}{k_1(s+b)}$, then

$$A\cos\int_0^s(-\tau)\text{ds}-\text{B}\sin\int_0^s(-\tau)\text{ds}=\kappa^{-1}.$$

Since cosine function is an even function and sinus function is an odd function,

$$A\cos\int_0^s\tau\ \text{ds}+\text{B}\sin\int_0^s\tau\ \text{ds}=\kappa^{-1}.$$

Consequently, $\beta(s)$ is a timelike spherical curve.

Conversely, let $\beta(s)$ be a timelike spherical curve having $\tau=-k_3$, $\kappa=\frac{k_2}{k_1(s+b)}$. Then,

$$(A\cos\int_0^s\tau\ \text{ds}+\text{B}\sin\int_0^s\tau\ \text{ds})=\kappa^{-1}(s).$$

Substituting $\tau=-k_3$, $\kappa=\frac{k_2}{k_1(s+b)}$ and doing the necessary calculations, we get

$$(A\cos\int_0^s(-k_3)\text{ds}+\text{B}\sin\int_0^s(-k_3)\text{ds})=\frac{(s+b)k_1}{k_2}.$$

Since cosine and sinus functions are even and odd functions, respectively,

$$(A\cos\int_0^s k_3\ \text{ds}-\text{B}\sin\int_0^s k_3\ \text{ds})=\frac{(s+b)k_1}{k_2}.$$

As a result, it can be seen that α is a timelike rectifying curve.

Theorem 4.13. Let $\alpha=\alpha(s)$ be a spacelike curve having the curvatures k_1, k_2 and k_3 in E_1^4. And let $\beta=\beta(s)$ be a spacelike curve in E_1^3 having $\tau=k_3$, $\kappa=-\frac{k_2}{k_1(s+b)}$. Hence, $\alpha=\alpha(s)$ is a spacelike rectifying curve in E_1^4

(according to **Case 2.**) if and only if $\beta = \beta(s)$ is a spacelike spherical curve in E_1^3.

Proof. The proof can be given like the proof of Theorem (4.12).

Theorem 4.14. Let $\alpha = \alpha(s)$ be a spacelike curve having the curvatures k_1, k_2 and k_3 in E_1^4. And let $\beta = \beta(s)$ be a spacelike curve in E_1^3 having $\tau = -k_3$, $\kappa = -\frac{k_2}{k_1(s+b)}$. Hence, $\alpha = \alpha(s)$ is a spacelike rectifying curve in E_1^4 (according to **Case 3.**) if and only if $\beta = \beta(s)$ is a spacelike spherical curve in E_1^3.

Proof. The proof can be given like the proof of Theorem (4.12).

Now, these theorems can be generalized to E_1^n.

Theorem 4.15. Let α be a timelike curve in E_1^n and let $\{T, N, N_1, N_2, \dots, N_{n-2}\}$ be an RMF on the direction curve $\gamma = \int N(s)\, ds$. α is a timelike rectifying curve in E_1^n if and only if

$$\mu_1 \bar{k}_1 - \left(\sum_{i=2}^{n-2} \mu_i \bar{k}_i\right) = (s+b)k_1,$$

where k_1 and $\bar{k}_i$ are the Rotation minimizing curvatures for $i = 1, 2, \dots, n-1$ and μ_i are constants for $i = 1, \dots, n$.

Proof. $(\Longrightarrow)$ Let $\alpha = \alpha(s)$ be a timelike rectifying curve in E_1^n. Then,

$$\alpha(s) = \lambda_1 T + \mu_1 N_1 + \mu_2 N_2 + \dots + \mu_{n-2} N_{n-2}.$$

Taken derivatives of both sides and substituting $T' = k_1 N$, $N'_1 = -\bar{k}_1 N$, $N'_2 = \bar{k}_2 N, \dots,$ $N'_{n-2} = \bar{k}_{n-2} N$, we get

$$T = \alpha' = \lambda'_1 T + (\lambda_1 k_1 - \mu_1 \bar{k}_1 + \mu_2 \bar{k}_2 + \dots + \mu_{n-2} \bar{k}_{n-2})N + \mu'_1 N_1 + \mu'_2 N_2 + \dots + \mu'_{n-2} N_{n-2}.$$

Doing the necessary calculations, we get

$$\lambda'_1 = 1, \mu'_1 = 0, \mu'_2 = 0, \dots, \mu'_{n-2} = 0,$$

and

$$\lambda_1 k_1 - \mu_1 \bar{k}_1 + \mu_2 \bar{k}_2 + \dots + \mu_{n-2} \bar{k}_{n-2} = 0.$$

Consequently, since $\lambda'_1 = 1$ and $\mu'_1 = 0$, $\mu'_2 = 0, \dots, \mu'_{n-2} = 0$, it is obtained that $\lambda_1 = s + b$, $\mu_1, \mu_2, \dots, \mu_{n-2} \in R$. Hence,

$$\mu_1 \bar{k}_1 - \mu_2 \bar{k}_2 - \dots - \mu_{n-2} \bar{k}_{n-2} = (s+b)k_1.$$

$(\Longleftarrow)$ Let

$$\mu_1 \bar{k}_1 - \mu_2 \bar{k}_2 - \dots - \mu_{n-2} \bar{k}_{n-2} = (s+b)k_1,$$

where $\mu_1, \mu_2, \dots, \mu_{n-2} \in R$ and k_1, $\bar{k}_i$ are the Rotation minimizing curvatures for $i = 1, 2, \dots, n-1$. Since

$$\frac{d}{ds}(\alpha(s) - (s+b)T - \mu_1 N_1 - \mu_2 N_2 - \dots - \mu_{n-2} N_{n-2}) = 0,$$

$\alpha(s) = (s+b)T + \mu_1 N_1 + \mu_2 N_2 + \dots + \mu_{n-2} N_{n-2} + \vec{C}$ is a timelike rectifying curve in E_1^n. Thus, the proof is completed.

Theorem 4.16. Let α be a spacelike curve in E_1^n and let $\{T, N, N_1, N_2, \dots, N_{n-2}\}$ be an RMF on the direction curve $\gamma = \int N(s)\, ds$. α is a spacelike rectifying curve in E_1^n (according to **Case 2.**) if and only if

$$\left(\sum_{i=1}^{n-2} \mu_i \bar{k}_i\right) = (s+b)k_1,$$

where k_1 and $\bar{k}_i$ are the Rotation minimizing curvatures for $i = 1, 2, \dots, n-1$ and μ_i are constants for $i = 1, \dots, n$.

Proof. The proof can be given like the proof of Theorem (4.15).

Theorem 4.17. Let α be a spacelike curve in E_1^n and let $\{T, N, N_1, N_2, \dots, N_{n-2}\}$ be an RMF on the direction curve $\gamma = \int N(s)\, ds$. α is a spacelike rectifying curve in E_1^n (according to **Case 3.**) if and only if

$$-\mu_1 \bar{k}_1 + \left(\sum_{i=2}^{n-2} \mu_i \bar{k}_i\right) = (s+b)k_1,$$

where k_1 and $\bar{k}_i$ are the Rotation minimizing curvatures for $i = 1, 2, \dots, n-1$ and μ_i are constants for $i = 1, \dots, n$.

Proof. The proof can be given like the proof of Theorem (4.15).

Theorem 4.18. Let $\alpha = \alpha(s)$ be a unit speed timelike rectifying curve in E_1^n with non-zero the Rotation minimizing curvatures $\bar{k}_1$, $\bar{k}_2$ and k_1. Then, the following statements hold;

- $\langle \alpha(s), T \rangle = -(s+b)$, *i.e.* the curve is tangential.
- The distance function $\rho = \|\alpha(s)\|$ satisfies

$$\|\alpha(s)\|^2 = -(s+b)^2 + \sum_{i=1}^{n-2} \mu_i{}^2 = -s^2 + as + c,$$

where $a, b, c, \mu_1, \mu_2, \dots, \mu_{n-2} \in R$.

- $\langle \alpha(s), N_i \rangle = \mu_i, i = 1,2, \dots, n-2, \mu_i \in R$.

Proof.

- $\langle \alpha(s), T \rangle = (s+b)\langle T, T \rangle + \mu_1 \langle N_1, T \rangle + \mu_2 \langle N_2, T \rangle + \dots + \mu_{n-2} \langle N_{n-2}, T \rangle$. Since $\langle T, T \rangle = -1$ and $\langle N_1, T \rangle = \langle N_2, T \rangle = \dots = \langle N_{n-2}, T \rangle = 0$, we get $\langle \alpha(s), T \rangle = -(s+b)$. Thus, the curve is tangential.
- The proof is obvious from the inner product,

$$\langle \alpha(s), \alpha(s) \rangle = \|\alpha(s)\|^2 = -(s+b)^2 + \sum_{i=1}^{n-2} \mu_i{}^2,$$

where $b, \mu_1, \mu_2, \dots, \mu_{n-2} \in R$.

- Since $\langle \alpha(s), N_1 \rangle = (s+b)\langle T, N_1 \rangle + \mu_1 \langle N_1, N_1 \rangle + \mu_2 \langle N_2, N_1 \rangle + \dots + \mu_{n-2} \langle N_{n-2}, N_1 \rangle$ and $\langle T, N_1 \rangle = \langle N_2, N_1 \rangle = \dots = \langle N_{n-2}, N_1 \rangle = 0$, $\langle N_1, N_1 \rangle = 1$, we get $\langle \alpha(s), N_1 \rangle = \mu_1 = constant$. Also, we get $\langle \alpha(s), N_2 \rangle = \mu_2 = constant, \dots, \langle \alpha(s), N_{n-2} \rangle = \mu_{n-2} = constant$.

Theorem 4.19. Let $\alpha = \alpha(s)$ be a unit speed spacelike rectifying curve in E_1^n with non-zero the Rotation minimizing curvatures $\bar{k}_1$, $\bar{k}_2$ and k_1 (according to **Case 2.**). Then, the following statements hold;

- $\langle \alpha(s), T \rangle = (s+b)$, i.e. the curve is tangential.
- The distance function $\rho = \|\alpha(s)\|$ satisfies

$$\|\alpha(s)\|^2 = (s+b)^2 + \mu_1{}^2 - \sum_{i=2}^{n-2} \mu_i{}^2 = s^2 + as + c,$$

where $a, b, c, \mu_1, \mu_2, \dots, \mu_{n-2} \in R$.

- $\langle \alpha(s), N_i \rangle = \mu_i, i = 1,2, \dots, n-2, \mu_i \in R$.

Proof.

- $\langle \alpha(s), T \rangle = (s+b)\langle T, T \rangle + \mu_1 \langle N_1, T \rangle + \mu_2 \langle N_2, T \rangle + \dots + \mu_{n-2} \langle N_{n-2}, T \rangle$. Since $\langle T, T \rangle = 1$ and
- $\langle N_1, T \rangle = \langle N_2, T \rangle = \dots = \langle N_{n-2}, T \rangle = 0$, we get $\langle \alpha(s), T \rangle = (s+b)$. Thus, the curve is tangential.
- The proof is obvious from the inner product,

$$\langle \alpha(s), \alpha(s) \rangle = \|\alpha(s)\|^2 = (s+b)^2 + \mu_1{}^2 - \sum_{i=2}^{n-2} \mu_i{}^2 = s^2 + as + c,$$

where $b, \mu_1, \mu_2, \dots, \mu_{n-2} \in R$.

- Since $\langle \alpha(s), N_1 \rangle = (s+b)\langle T, N_1 \rangle + \mu_1 \langle N_1, N_1 \rangle + \mu_2 \langle N_2, N_1 \rangle + \dots + \mu_{n-2} \langle N_{n-2}, N_1 \rangle$ and $\langle T, N_1 \rangle = \langle N_2, N_1 \rangle = \dots = \langle N_{n-2}, N_1 \rangle = 0$, $\langle N_1, N_1 \rangle = 1$, we get $\langle \alpha(s), N_1 \rangle = \mu_1 = constant$. Also, we get $\langle \alpha(s), N_2 \rangle = -\mu_2 = constant, \dots, \langle \alpha(s), N_{n-2} \rangle = -\mu_{n-2} = constant$.

Theorem 4.20. Let $\alpha = \alpha(s)$ be a unit speed spacelike rectifying curve in E_1^n with non-zero the Rotation minimizing curvatures $\bar{k}_1$, $\bar{k}_2$ and k_1 (according to **Case 3.**). Then, the following statements hold;

- $\langle \alpha(s), T \rangle = (s+b)$, i.e. the curve is tangential.
- The distance function $\rho = \|\alpha(s)\|$ satisfies

$$\|\alpha(s)\|^2 = (s+b)^2 + \sum_{i=1}^{n-2} \mu_i{}^2 = s^2 + as + c,$$

where $a, b, c, \mu_1, \mu_2, \dots, \mu_{n-2} \in R$.

- $\langle \alpha(s), N_i \rangle = \mu_i, i = 1,2, \dots, n-2, \mu_i \in R$.

Proof.

- $\langle \alpha(s), T \rangle = (s+b)\langle T, T \rangle + \mu_1 \langle N_1, T \rangle + \mu_2 \langle N_2, T \rangle + \dots + \mu_{n-2} \langle N_{n-2}, T \rangle$. Since $\langle T, T \rangle = 1$ and $\langle N_1, T \rangle = \langle N_2, T \rangle = \dots = \langle N_{n-2}, T \rangle = 0$, we get $\langle \alpha(s), T \rangle = (s+b)$. Thus, the curve is tangential.
- The proof is obvious from the inner product,

$$\langle \alpha(s), \alpha(s)\rangle = \|\alpha(s)\|^2 = (s+b)^2 + \sum_{i=1}^{n-2} {\mu_i}^2 = s^2 + as + c,$$

where $b, \mu_1, \mu_2, \dots, \mu_{n-2} \in R$.

- Since $\langle \alpha(s), N_1\rangle = (s+b)\langle T, N_1\rangle + \mu_1 \langle N_1, N_1\rangle + \mu_2 \langle N_2, N_1\rangle + \cdots + \mu_{n-2} \langle N_{n-2}, N_1\rangle$ and $\langle T, N_1\rangle = \langle N_2, N_1\rangle = \cdots = \langle N_{n-2}, N_1\rangle = 0$, $\langle N_1, N_1\rangle = 1$, we get $\langle \alpha(s), N_1\rangle = \mu_1 = constant$. Also, we get $\langle \alpha(s), N_2\rangle = \mu_2 = constant, \dots,$ $\langle \alpha(s), N_{n-2}\rangle = \mu_{n-2} = constant$.

Result 4.4. The coefficients μ_i are constants for $\alpha(s) = (s+b)T + \mu_1 N_1 + \cdots + \mu_{n-2} N_{n-2}$. But, coefficients B_1, B_2 are functions in E^4, see [16]. Also, coefficients $B_1, B_2, \dots, B_{n-2}$ are functions in E^n, see [6].

5. Conclusion

In this paper, some applications of a Rotation minimizing frame (RMF) are studied in E_1^4 for timelike and spacelike curves. An RMF is obtained on the direction curve $\int \mathrm{N(s)\,ds}$ for timelike and spacelike curves in E_1^4. The condition of being a spherical curve is given with the help of this frame for timelike and spacelike curves. This frame is generalized to E_1^{n}. This frame is applied to a timelike and a spacelike rectifying curve in E_1^4 and in E_1^{n}. However, the coefficients of timelike rectifying curves are functions [6, 7], these coefficients of timelike and spacelike rectifying curves are constants in our paper.

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References

[1] Bishop, L. R. (1975). There is more than one way to frame a curve. Amer. Math. Monthly, 82(3): 246-251.

[2] Etayo, F. (2016). Rotation Minimizing Vector Fields and Frames in Riemannian Manifolds. Geometry, Algebra and Applications: From Mechanics to Cryptography, 161, 91-100.

[3] Etayo, F. (2018). Geometric Properties of RM vector field along curves in Riemannian Manifolds. Turkish Journal of Mathematics, 42, 121-130.

[4] Jüttler, B. (1998). Rotation Minimizing Spherical Motions. Advances in Robot Kinematics: Analysis and Control, pp. 413-422.

[5] Wang, W., Jüttler, B., Zheng, D., and Liu, Y. (2008). Computation of Rotation Minimizing Frame. ACM Transactions on Graphics, 27(1), Article No. 2: 18 pages.

[6] Cambie, S., Geomans, W., and Bussche, I. W. D. (2006). Rectifying curves in the n-dimensional Euclidean space. Turk. J. Math., 40: 210-223.

[7] Ilarslan, K., Nešović, E., and Petrović-Torgaṽev, M. (2003). Some characterizations of rectifying curves in the Minkowski 3-space. Novi Sad J. Math., Vol. 2003, 33(2), 23-32.

[8] López, R. (2014). Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space. Int. Electron. J. Geom., 7(1), 44-107.

[9] Walrave, J. (1995). Curves and Surfaces in Minkowski Space. K.U. LEUVEN Faculteit Der Wetenschappen (PHD Thesis). P. 147.

[10] Gökçelik, F., Bozkurt, Z., Gök, I., Ekmekci, F. N., and Yaylı, Y. (2014). Parallel Transport Frame in 4-dimensional Euclidean Space E^4. Caspian Journal of Mathematical Sciences (CJMS), 3(1), 91-103.

[11] Ahmad, T. A. and Önder, M. (2012). Some Characterizations of Rectifying Spacelike Curves in the Minkowski Space-Time. Global Journal of Science Frontier Research Mathematics & Decision Sciences, 12(1), p. 9.

[12] Wong, Yung-Chow. (1972). On explicit characterization of spherical curves. Proceedings of the American Mathematical Society, 34(1): 239-242.

[13] Petrović-Torgaṽev, M. and Šućurović, E. (2001). Some Characterizations of the Lorentzian Spherical Timelike and Null Curves. Matematıqkı Vesnık, 53, 21-27.

[14] Petrović-Torgaṽev, M. and Šućurović, E. (2000). Some Characterizations of the Lorentzian Spherical Spacelike Curves with the Timelike and the Null Principal Normal. Mathematica Moravıca., 4, 83-92.

[15] Chen, B. Y. (2003). When does the position vector of a space curve always lie in its rectifying plane? Amer. Math. Monthly, 110(2): 147-152.

[16] Ilarslan, K. and Nešović, E. (2008). Some characterizations of rectifying curves in the Euclidean space E^4. Turk. J. Math., 32, 21-30.