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Stability of Stochastic Differential Equations with Distributed and State-Dependent Delays

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Abstract

Stability of a linear stochastic differential equation with distributed and state-dependent delays is investigated. Sufficient conditions of asymptotic mean square stability are obtained via the general method of Lyapunov functionals construction and the method of linear matrix inequalities (LMIs). Numerical simulations illustrate the theoretical results and open a new unsolved problem of the obtained stability conditions improving.

Keywords

Stochastic differential equation, State-dependent delays, Asymptotic mean square stability, Linear matrix inequalities (LMIs), Unsolved problem

1. Introduction

Differential equations with delays that are depending on the system state, so called, state-dependent delays, are enough new but at the same time are enough popular in research both in the deterministic case [1, 5-9, 11, 12, 14, 15, 17-20, 22-29, 32-34] and in the stochastic case [2, 3, 21, 30, 36, 37]. However, it is necessary to note that research for stochastic differential equations with state-dependent delays, in particular, with problems of stability for equations of such type, one can meet much less common. In this paper, we try to improve a bit this situation and to show how can be obtained sufficient conditions of asymptotic mean square stability for the linear Ito stochastic differential equation [13] with distributed delays, depending on time and on the system state

$$\begin{gathered} dx(t)=\left(Ax(t)+\sum_{i=1}^{k}\beta_i(x_t)\right)dt+\sum_{i=1}^{m}\gamma_i(x_t)dw_i(t), \\ \beta_i(x_t)=\int_{t-h_i(t,x(t))}^{t}B_i x(s)ds, \quad \gamma_i(x_t)=\int_{t-\tau_i(t,x(t))}^{t}C_i x(s)ds, \\ x_0(s)=\phi(s)\in H_2, \quad s\le 0. \end{gathered} \tag{1.1}$$

Here, $x(t)\in\mathbf{R}^n$, $A,B_i,C_i\in\mathbf{R}^{n\times n}$, $w_1(t),\cdots,w_m(t)$, are mutually independent standard Wiener processes on a complete probability space $\{\Omega,\mathfrak{F},\mathbf{P}\}$, $\{\mathfrak{F}_t, t\ge 0\}$ is a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, *i.e.*, $\mathfrak{F}_{t_1}\subset\mathfrak{F}_{t_2}$, for $t_1 < t_2$, H_2 is a space of $\mathfrak{F}_0$-adapted stochastic processes $\phi(s)\in\mathbf{R}^n$, $s\le 0$, $\|\phi\|^2=\sup_{s\le 0}\mathbf{E}|\phi(s)|^2$, $\mathbf{E}$ is the mathematical expectation with respect to the measure $\mathbf{P}$ [13, 31].

Below via the general method of Lyapunov functionals construction [31] sufficient conditions for asymptotic mean square stability of the zero solution of the Eq. (1.1) are obtained in the terms of linear matrix inequalities (LMIs) [4, 10, 35].

1.1 Auailiary definitions and statements

Definition 1.1 The zero solution of the Eq. (1.1) is called:
- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t,\phi)|^2 < \varepsilon, t \geq 0$, proided that $\|\phi\|^2 < \delta$;
- asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty}\mathbf{E}\left|x(t,\phi)\right|^2 = 0$ for each initial function ϕ.

Let $x(t)$ be a value of the solution of the Eq. (1.1) in the time moment t, $x_t = x(t+s)$, $s < 0$, be the trajectory of the solution of the Eq. (1.1) until the time moment t. Consider a functional $V(t,\varphi):[0,\infty)\times H_2 \to \mathbf{R}_+$ that can be presented in the form $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s<0$, and for $\varphi = x_t$ put

$$V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+s)), \qquad x = \varphi(0) = x(t), \qquad s<0. \tag{1.2}$$

Denote by D the set of functionals, for which the function $V_\varphi(t,x)$, defined in (1.2), has a continuous derivative with respect to t and two continuous derivatives with respect to x. Let ' be the sign of transpose, ∇ and ∇^2 be respectively the first and the second derivatives of the function $V_\varphi(t,x)$ with respect to x. For the functionals from D the generator L of the Eq. (1.1) has the form [13, 31]

$$LV(t,x) = \frac{\partial V_\varphi(t,x(t))}{\partial t} + \nabla V_\varphi'(t,x(t))\left(Ax(t) + \sum_{i=1}^{k}\beta_i(x_t)\right) + \frac{1}{2}\sum_{i=1}^{m}\gamma_i'(x_t)\nabla^2 V_\varphi(t,x(t))\gamma_i(x_t). \tag{1.3}$$

Theorem 1.1 [31] Let there exist a functional $V(t,\varphi)\in D$, positive constants c_1, c_2, c_3, such that the following conditions hold:

$$\mathbf{E}V(t,x_t) \geq c_1\mathbf{E}\left|x(t)\right|^2, \qquad \mathbf{E}V(0,\phi) \leq c_2\|\phi\|^2, \qquad \mathbf{E}LV(t,x_t) \leq -c_3\mathbf{E}\left|x(t)\right|^2.$$

Then the zero solution of the Eq. (1.1) is asymptotically mean square stable.

Lemma 1.1 [31] For arbitrary vectors $a,b\in\mathbf{R}^n$ and a positive definite matrix $R\in\mathbf{R}^{n\times n}$ the following inequality holds: $a'b + b'a \leq a'Ra + b'R^{-1}b$.

Schur complement [16]. The symmetric matrix $\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$, where $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times m}$, $C\in\mathbf{R}^{m\times m}$, is negative definite if and only if C and $A - BC^{-1}B'$ are both negative definite.

2. Main result

We will assume that the delays $h_i(t,x)$ and $\tau_i(t,x)$ are bounded, *i.e.*, satisfy the conditions:

$$\begin{aligned} &0 \leq h_i(t,x) \leq h_i, \quad i = 1,\cdots,k, \\ &0 \leq \tau_i(t,x) \leq \tau_i, \quad i = 1,\cdots,m. \end{aligned} \tag{2.1}$$

Note that this assumption includes particular cases of constant delays (h_i,τ_i) and bounded time-varying delays $(h_i(t),\tau_i(t))$.

Theorem 2.1 Let the conditions (2.1) hold and there exist positive definite matrices $P, R_{11},\cdots,R_{1k},R_{21},\cdots,R_{2m}\in\mathbf{R}^{n\times n}$ that satisfy the *LMI*

$$\begin{bmatrix} \Phi & \Psi \\ \Psi' & R \end{bmatrix} < 0, \qquad \begin{aligned} &\Phi = A'P + PA + \sum_{i=1}^{k} h_i R_{1i} + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 C_i' R_{2i} C_i \in \mathbf{R}^{n\times n}, \\ &\Psi = \left[\sqrt{h_1}B_1'P \cdots \sqrt{h_k}B_k'P\,\frac{\tau_1}{\sqrt{2}}C_1'P \cdots \frac{\tau_m}{\sqrt{2}}C_m'P\right] \in \mathbf{R}^{n\times(k+m)n}, \\ &R = diag\{-R_{11},\cdots,-R_{1k},-R_{21},\cdots,-R_{2m}\} \in R^{(k+m)n\times(k+m)n}. \end{aligned} \tag{2.2}$$

Then the zero solution of the Eq. (1.1) is asymptotically mean square stable.

Proof: Following the general method of Lyapunov functionals construction [31], we will construct the Lyapunov functional V in the form $V = V_1 + V_2$, where $V_1(x(t)) = x'(t)Px(t)$ and V_2 will be chosen below. Via (1.3) we have

$$\begin{aligned} LV_1(x(t)) &= 2x'(t)P\left(Ax(t) + \sum_{i=1}^{k}\beta_i(x_t)\right) + \sum_{i=1}^{m}\gamma_i'(x_t)P\gamma_i'(x_t) \\ &= x'(t)(A'P + PA)x(t) + \sum_{i=1}^{k} I_{1i} + \sum_{i=1}^{m} I_{2i}, \end{aligned} \tag{2.3}$$

where via (1.1)

$$I_{1i} = \int_{t-h_{i(t,x(t))}}^{t}\left(x'(t)PB_i x(s) + x'(s)B_i'Px(t)\right)ds,$$

$$I_{2i} = \int_{t-\tau_{i(t,x(t))}}^{t}\int_{t-\tau_{i(t,x(t))}}^{t} x'(s)C_i'PC_i x(\theta)\,d\theta ds.$$

Using Lemma 1.1 for I_{1i} with $R_{1i} > 0$, $a = x(t)$, $b = PB_i x(s)$, via (2.1) we have

$$\begin{aligned} I_{1i} &\le \int_{t-h_i(t,x(t))}^{t}\left(x'(t)R_{1i}x(t) + x'(s)B_i'PR_{1i}^{-1}PB_i x(s)\right)ds \\ &\le h_i x'(t)R_{1i}x(t) + \int_{t-h_i}^{t} x'(s)B_i'PR_{1i}^{-1}PB_i x(s)ds. \end{aligned} \tag{2.4}$$

Similarly, using Lemma 1.1 for I_{2i} with $R_{2i} > 0$, $a = C_i x(s)$, $b = PC_i x(0)$, via (2.1) we obtain

$$\begin{aligned} I_{2i} &= \frac{1}{2}\int_{t-\tau_i(t,x(t))}^{t}\int_{t-\tau_i(t,x(t))}^{t}\left(x'(s)C_i'PC_i x(\theta) + x'(\theta)C_i'PC_i x(s)\right)d\theta ds \\ &\le \frac{1}{2}\int_{t-\tau_i(t,x(t))}^{t}\int_{t-\tau_i(t,x(t))}^{t}\left(x'(s)C_i'R_{2i}C_i x(s) + x'(\theta)C_i'PR_{2i}^{-1}PC_i x(\theta)\right)d\theta ds \\ &\le \frac{1}{2}\tau_i\int_{t-\tau_i}^{t} x'(s)\left(C_i'R_{2i}C_i + C_i'PR_{2i}^{-1}PC_i\right)x(s)ds. \end{aligned} \tag{2.5}$$

From (2.3), (2.4), (2.5) it follows that

$$\begin{aligned} LV_1(x(t)) &\le x'(t)\left(A'P + PA + \sum_{i=1}^{k}h_i R_{1i}\right)x(t) + \sum_{i=1}^{k}\int_{t-h_i}^{t} x'(s)B_i'PR_{1i}^{-1}PB_i x(s)ds \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\tau_i\int_{t-\tau_i}^{t} x'(s)C_i'\left(R_{2i} + PR_{2i}^{-1}P\right)C_i x(s)ds. \end{aligned} \tag{2.6}$$

Choosing the additional functional V_2 in the form

$$\begin{aligned} V_2(x_t) &= \sum_{i=1}^{k}\int_{t-h_i}^{t}(s - t + h_i)x'(s)B_i'PR_{1i}^{-1}PB_i x(s)ds \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\tau_i\int_{t-\tau_i}^{t}(s - t + \tau_i)x'(s)C_i'\left(R_{2i} + PR_{2i}^{-1}P\right)C_i x(s)ds, \end{aligned}$$

we obtain

$$\begin{aligned} LV_2(x_2) &= \sum_{i=1}^{k}h_i x'(t)B_i'PR_{1i}^{-1}PB_i x(t) - \sum_{i=1}^{k}\int_{t-h_i}^{t} x'(s)B_i'PR_{1i}^{-1}PB_i x(s)ds \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 x'(t)C_i'\left(R_{2i} + PR_{2i}^{-1}P\right)C_i x(t) - \frac{1}{2}\sum_{i=1}^{m}\tau_i\int_{t-\tau_i}^{t} x'(s)C_i'\left(R_{2i} + PR_{2i}^{-1}P\right)C_i x(s)ds. \end{aligned} \tag{2.7}$$

From (2.6), (2.7) for the functional $V = V_1 + V_2$ we have

$$LV(x_t) \le x'(t)Qx(t),$$

$$Q = A'P + PA + \sum_{i=1}^{k} h_i\left(R_{1i} + B_i'PR_{1i}^{-1}PB_i\right) + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 C_i'\left(R_{2i} + PR_{2i}^{-1}P\right)C_i.$$

If the matrix Q is negative definite then the constructed functional $V(x_t)$ satisfies all conditions of Theorem 1.l and therefore the zero solution of the Eq. (1.1) is asymptotically mean square stable.

Using the matrices Φ, Ψ and R, defined in (2.2), one can represent the matrix Q in the form

$$\begin{aligned} Q &= A'P + PA + \sum_{i=1}^{k} h_i R_{1i} + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 C_i'R_{2i}C_i + \sum_{i=1}^{k} h_i B_i'PR_{1i}^{-1}PB_i + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 C_i'PR_{2i}^{-1}PC_i \\ &= \Phi - \Psi R^{-1}\Psi'. \end{aligned} \tag{2.8}$$

Via Schur complement the matrix Q is negative definite if and only if the LMI (2.2) holds. The proof is completed.

Remark 2.1 Using Lemma 1.1 with another representation for a and b it is possible to get another *LMI* in Theorem 2.1. For example, using $a = Px(t)$, $b = B_i x(s)$ for I_{1i} and $a = x(s)$, $b = C_i'PC_i x(0)$ for I_{2i}, we have

$$I_{1i} \le h_i x'(t)PR_{1i}Px(t) + \int_{t-h_i}^{t} x'(s)B_i'R_{1i}^{-1}B_i x(s)ds,$$

$$I_{2i} \le \frac{\tau_i}{2}\int_{t-\tau_i}^{t} x'(s)\left(R_{2i} + C_i'PC_iR_{2i}^{-1}C_i'PC_i\right)x(s)ds.$$

Choosing the additional functional V_2 in the form

$$\begin{aligned} V_2(x_t) &= \sum_{i=1}^{k}\int_{t-h_i}^{t}(s-t+h_i)x'(s)B_i'R_{1i}^{-1}B_i x(s)ds \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\tau_i\int_{t-\tau_i}^{t}(s-t+\tau_i)x'(s)\left(R_{2i} + C_i'PC_iR_{2i}^{-1}C_i'PC_i\right)x(s)ds, \end{aligned}$$

instead of (2.8) and (2.2) we obtain the matrix Q and the *LMI* in another form

$$\begin{aligned} Q &= A'P + PA + \sum_{i=1}^{k} h_i PR_{1i}P + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 R_{2i} + \sum_{i=1}^{k} h_i B_i'R_{1i}^{-1}B_i + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 C_i'PC_iR_{2i}^{-1}C_i'PC_i \\ &= \Phi - \Psi R^{-1}\Psi', \end{aligned}$$

$$\begin{bmatrix} \Phi & \Psi \\ \Psi' & R \end{bmatrix} < 0, \quad \begin{aligned} \Phi &= A'P + PA + \sum_{i=1}^{k} h_i PR_{1i}P + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 R_{2i} \in \mathbf{R}^{n\times n}, \\ \Psi &= \left[\sqrt{h_1}B_1' \cdots \sqrt{h_k}B_k' \ \frac{\tau_1}{\sqrt{2}}C_1'PC_1 \cdots \frac{\tau_m}{\sqrt{2}}C_m'PC_m\right] \in \mathbf{R}^{n\times(k+m)n}, \\ R &= diag\{-R_{11}, \cdots, -R_{1k}, -R_{21}, \cdots, -R_{2m}\} \in \mathbf{R}^{(k+m)n\times(k+m)n}. \end{aligned} \tag{2.9}$$

Using other different representations for a and b, one can get other different forms of the *LMI* in Theorem 2.1.

Remark 2.2 In the scalar case both *LMIs* (2.2) and (2.9) hold if and only if $A + \sum_{i=1}^{k} h_i|B_i| + \frac{1}{2}\sum_{i=1}^{m}\tau_i^2 C_i^2 < 0$.

3. Numerical examples

Here we consider two cases of the stochastic differential equation

$$dx(t) = \left(Ax(t) + \int_{t-h(x(t))}^{t} Bx(s)ds\right)dt + \int_{t-\tau}^{t} Cx(s)dsdw(t). \tag{3.1}$$

3.1. System of two equations

Suppose that in the Eq. (3.1) $0 \le h(x) \le h$, τ is a constant and

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.2}$$

The *LMI* (2.2) takes the form

$$\begin{bmatrix} A'P+PA+hR_1+\frac{1}{2}\tau^2C'R_2C & \sqrt{h}B'P & \frac{\tau}{\sqrt{2}}C'P \\ \sqrt{h}PB & -R_1 & 0 \\ \frac{\tau}{\sqrt{2}}PC & 0 & -R_2 \end{bmatrix} < 0. \tag{3.3}$$

Via MATLAB it was shown that for the matrices (3.2) there exist positive definite matrices P, R_1, $R_2 \in \mathbf{R}^{2\times 2}$ such that the LMI (3.3) holds, for instance, for $h = 1$, $\tau = 0.1$ or for $h = \tau = 0.73$ or for $h = 0.49$, $\tau = 1$. So, for these values of the delays the zero solution of the Eq. (3.1), (3.2) is asymptotically mean square stable.

3.2. *A* scalar equation

Consider the Eq. (3.1) in the scalar case. Via Remark 2.2, the inequality $\lambda = A + h|B| + \frac{1}{2}\tau^2C^2 < 0$ is a sufficient condition for asymptotic mean square stability of the zero solution of the Eq. (3.1).

In Figure 1, 30 trajectories of a solution of the Eq. (3.1) are shown for the following values of the parameters $A = -0.55$, $B = 0.4$, $C = -5$, $\tau = 0.1$, $h(x) = \frac{1}{1+x^2}$, and the initial function $\phi(s) = 2\cos(s)$, $s \in [-1, 0]$. In this case, the stability condition $\lambda = -0.025 < 0$ holds, therefore, the zero solution of the Eq. (3.1) is asymptotically mean square stable and all trajectories converge to zero.

Note that the delay $h(x) = \frac{1}{1+x^2}$ satisfies the conditions $h(x) \le 1$ and $\lim_{x\to 0} h(x) = 1$. In Figure 2, 30 trajectories of a solution of the Eq. (3.1) are shown for the same values of the parameters A, B, C, τ and for the delay $h(x) = \frac{x^2}{1+x^2}$ that satisfies the conditions $h(x) \le 1$ and $\lim_{x\to 0} h(x) = 0$.

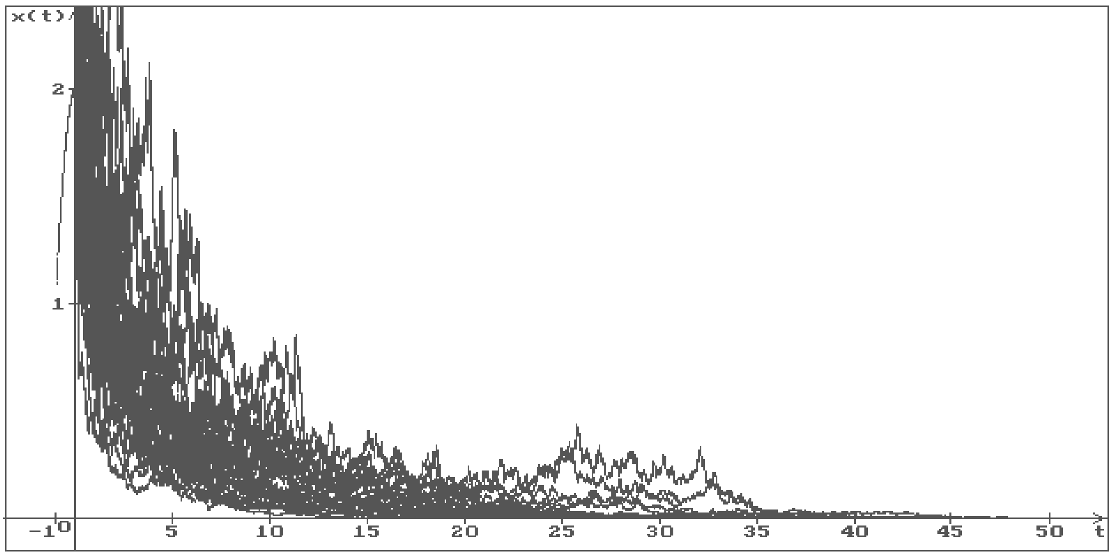

Figure 1. 30 trajectories of the equation (3.1) solution: $A = -0.55$, $B = 0.4$, $C = -5$, $\tau = 0.1$, $h(x) = \frac{1}{1+x^2}$

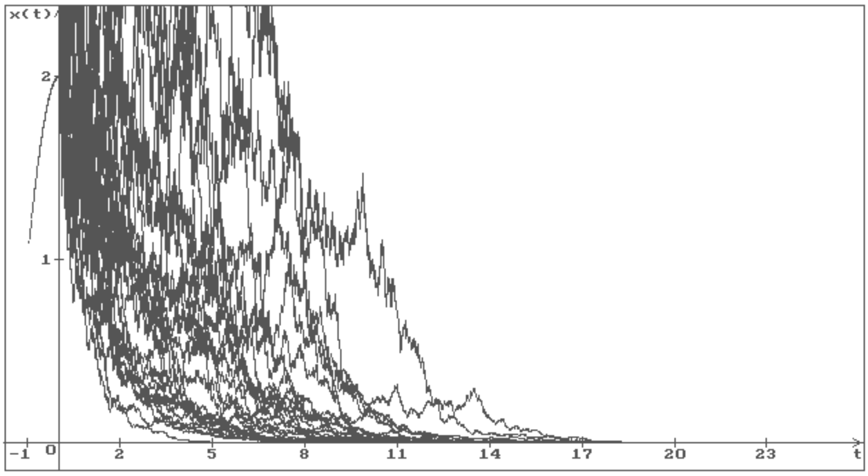

Figure 2. 30 trajectories of the equation (3.1) solution: $A=-0.55, \quad B=0.4, \quad C=-5, \quad \tau=0.1, \quad h(x)=\frac{x^2}{1+x^2}$

As in the previous case the stability condition $\lambda=-0.025<0$ holds and all trajectories converge to zero. However, one can see that the trajectories of a solution converge to zero essentially quicker than in Figure 1. This is explained by the fact that in the first case, when $x(t)$ tends to zero the delay $h(x(t))$ increases and tends to one, but in the second case, when the $x(t)$ tends to zero, the delay $h(x(t))$ decreases to zero too. Inspite of the fact that in the both cases the condition $h(x)\le 1$ holds it means that the obtained sufficient stability condition takes into account only the maximum value of the delay but does not take into account the particularity of its behavior.

Consider also Figure 3, where 30 trajectories of a solution of the Eq. (3.1) are shown for the same values of all parameters as in Figure 2 except of $A=-0.5$. In this case $\lambda=0.025>0$, i.e., the stability condition $\lambda<0$ does not hold. Despite of this all trajectories of a solution of the Eq. (3.1) converge to zero too although not as fast as in Figure 2. This fact emphasizes that the obtained stability condition is a sufficient one only and in principle can be improved.

So, we obtain the following

Unsolved problem. To get conditions of asymptotic mean square stability for the zero solution of the Eq. (1.1) that take into account the form of dependence of delays on the system state.

Remark 3.1 Note that by numerical simulation of the Eq. (3.1) solution for numerical simulation of the Wiener process trajectories the algorithm described in [31] was used here.

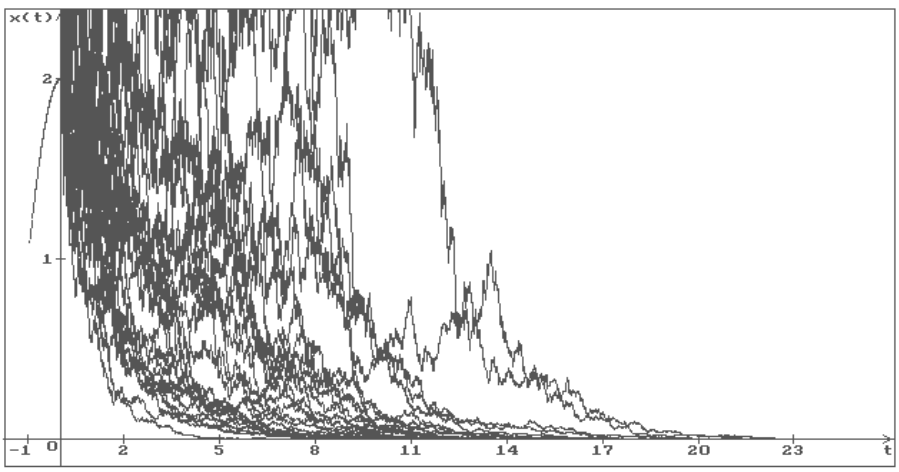

Figure 3. 30 trajectories of the equation (3.1) solution: $A=-0.5, \quad B=0.4, \quad C=-5, \quad \tau=0.1, \quad h(x)=\frac{x^2}{1+x^2}$

4. Conclusions

New conditions of asymptotic mean square stability are obtained for a linear stochastic differential equation with distributed and state-dependent delays. Numerical simulations illustrate the obtained results. For future investigation an unsolved problem related to the improvement of the obtained stability conditions is also proposed.

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