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A New Approach Comparison of the Farthest Point Map in Fuzzy and Classic N-Normed Spaces with Examples

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Abstract

In this paper, we have studied comparison of the farthest point map in different normed spaces with examples. First of all, we have given some definitions and theorems. Then we have compared with examples using the definitions we gave earlier and we show that the farthest point maps and farthest point sets are equal in different normed spaces.

Keywords

Farthest point, Farthest point map, Farthest point set, Fuzzy norm, Fuzzy n-norm, Fuzzy farthest point map.

1. Introduction and Preliminaries

Bag and Samanta [1] introduced a definition of a fuzzy norm and proved a decomposition theorem of a fuzzy norm into a family of crisp norms. Also, Bag and Samanta [2] give some properties on fuzzy norms. The concept of 2-norm and n -norm on a linear space has been introduced and developed by Gähler in [4, 5]. Following Misiak [10] and Malčeski [9] developed the theory of n -normed space.

Narayanan and Vijayabalaji [7] introduced the concept of fuzzy n -normed linear space. Vijayabalaji and Thillaigovindan [8] introduced the notion of Cauchy sequence and convergent sequence in fuzzy n -normed linear space and studied the completeness of the fuzzy n -normed linear space. Many authors studied on fuzzy n -normed linear space.

Many mathematicians such as Asplund [16,17], Panda and Kapoor [12], Govindarajulu [14], Elumalai and Vijayaragavan [13], Saravanan and Vijayaragavan [15] etc. studied on farthest point and remotal sets. Recently,Mirmostafaee and Mirzavaziri [3], studied on the closability of farthest point map in fuzzy normed spaces and proved several theorems pertaining to this map. Then Turkmen and Efe [11] studied the same subject in fuzzy n-normed spaces. In this paper we prove that the farthest point map and farthest point set in fuzzy and crisp normed spaces is similar by using definition and theorem in the sense of [11].

First, we recall some definitions, theorem and example on $n-$ norm, fuzzy $n-$ norm, farthest point and other notion which used later (see [6],[7],[8],[11],[12],[13],[14],15]).

Definition 1 ([6]). Let $n\in N$ and X be a real vector space of dimension $d\geq n$. A real-valued function $\|\cdot,...,\cdot\|$ on $\underbrace{X\times\cdots\times X}_{n}$ satisfying the following four properties,

(1) $\|x_1,x_2,...,x_n\|=0$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,

(2) $\|x_1,x_2,...,x_n\|$ is invariant under any permutation,

(3) $\|x_1,x_2,...,\alpha x_n\|=|\alpha|.\|x_1,x_2,...,x_n\|$ for any $\alpha\in R$,

(4) $\|x_1,x_2,...,x_{n-1},y+z\|\leq\|x_1,x_2,...,x_{n-1},y\|+\|x_1,x_2,...,x_{n-1},z\|$,

is called an $n-$norm on X and the pair $(X,\|\cdot,...,\cdot\|)$ is called an $n-$normed space, where d is infinite.

Example 1. Let $X=R^n$ and $\|x_1,x_2,...,x_n\|_E=\text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn}\end{vmatrix}\right)$,

where $x_i=(x_{i1},...,x_{in})\in R^n$ for each $i=1,2,...,n$. Then $(X,\|\cdot,...,\cdot\|_E)$ is an $n-$normed space which is called Euclidean $n-$normed space.

Definition 2 ([7]). Let X be a linear space over a real field F. A fuzzy subset N of $\underbrace{X\times\cdots\times X}_{n}\times R$ (R set of real numbers) is called a fuzzy $n-$norm on X if and only if

(N1) for all $t\in R$ with $t\leq 0$, $N(x_1,x_2,...,x_n,t)=0$,

(N2) for all $t\in R$ with $t>0$, $N(x_1,x_2,...,x_n,t)=1$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,

(N3) $N(x_1,x_2,...,x_n,t)$ is invariant under any permutation of $x_1,x_2,...,x_n$,

(N4) for all $t\in R$ with $t>0$, $N(x_1,x_2,...,cx_n,t)=N(x_1,x_2,...,x_n,t/|c|)$ if $c\neq 0$, $c\in F$,

(N5) for all $s,t\in R$, $N(x_1,x_2,...,x_n+x'_n,s+t)\geq\min\{N(x_1,x_2,...,x_n,s),N(x_1,x_2,...,x'_n,t)\}$,

(N6) $N(x_1,x_2,...,x_n,\cdot)$ is a nondecreasing function of R and $\lim_{t\to\infty}N(x_1,x_2,...,x_n,t)=1$.

Then (X,N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 1 ([7]). From (N3), it follows that in a f-n-NLS, (N4) for all $t\in R$ with $t>0$, $N(x_1,x_2,...,cx_i,...,x_n,t)=N(x_1,x_2,...,x_i,...,x_n,t/|c|)$,

if $c\neq 0$, (N5) for all $s,t\in R$, $N(x_1,x_2,...,x_i+x'_i,...,x_n,s+t)\geq\min\{N(x_1,x_2,...,x_i,...,x_n,s),N(x_1,x_2,...,x'_i,...,x_n,t)\}$.

Example 2. Let $(X,\|\cdot,...,\cdot\|)$ be a $n-$normed space as in Definition 1. Define,

$$N(x_1,x_2,...,x_n,t)=\begin{cases}\dfrac{t}{t+\|x_1,x_2,...,x_n\|} & \text{if} \quad t>0, t\in R\\ 0 & \text{if} \quad t\le 0\end{cases}$$

for all $x_1,x_2,...,x_n\in X$. Then, (X,N) is a f-n-NLS.

Example 3. Let $(X,\|\cdot,...,\cdot\|)$ be a $n-$normed linear space and $\alpha,\beta>0$. Then,

$$N(x_1,x_2,...,x_n,t)=\begin{cases}\dfrac{\alpha t}{\alpha t+\beta\|x_1,x_2,...,x_n\|} & \text{if} \quad t>0, x_1,x_2,...,x_n\in X\\ 0 & \text{if} \quad t\le 0, x_1,x_2,...,x_n\in X\end{cases}$$

is a fuzzy $n-$norm on X.

Example 4. Let $(X,\|\cdot,...,\cdot\|)$ be a $n-$normed linear space and $\beta>\alpha>0$. Then,

$$N(x_1,x_2,...,x_n,t)=\begin{cases}0 & , \quad t\le\alpha\|x_1,x_2,...,x_n\|\\ \dfrac{t}{t+(\beta-\alpha)\|x_1,x_2,...,x_n\|} & , \quad \alpha\|x_1,x_2,...,x_n\|<t\le\beta\|x_1,x_2,...,x_n\|\\ 1 & , \quad t>\beta\|x_1,x_2,...,x_n\|\end{cases}$$

is a fuzzy $n-$norm on X.

Definition 3 ([8]). A sequence $\{x_k\}$ in a fuzzy n-normed space $(X,N,*)$ is said to converge to x if given $r>0$, $t>0$, $0<r<1$, there exists an integer $k_0\in N$ such that $N(x_1,x_2,...,x_{n-1},x_k-x,t)>1-r$, for all $k\ge k_0$.

Theorem 1 ([8]). In a fuzzy $n-$normed space $(X,N,*)$ a sequence $\{x_k\}$ converges to x if and only if $N(x_1,x_2,...,x_{n-1},x_k-x,t)\to 1$ as $k\to\infty$.

Definition 4 [11]. Let $\{x_k\}$ be a sequence in X and $\alpha\in(0,1)$. We say that the sequence $\{x_k\}$ is $\alpha-$convergent to $x\in X$, and write $x_k\to_\alpha x$ or $\alpha-\lim_{x\to\infty}x_k=x$ if $\forall\varepsilon>0, \forall\delta>0, \exists k_0$ such that $k\ge k_0$ implies that $N(x_1,x_2,...,x_{n-1},x_k-x,\delta)\ge\alpha-\varepsilon$. It is called convergent to x, and write $x_k\to x$, if it is $\alpha-$convergent to x for each $\alpha\in(0,1)$ or equivalently $\lim_{k\to\infty}N(x_1,x_2,...,x_{n-1},x_k-x,\varepsilon)=1$ for each $\varepsilon>0$.

Definition 5 [11]. Let A be a subset of X. A is said to be fuzzy $\alpha-$bounded if there is a positive real number m such that $N(a_1,a_2,...,a_n,m)\ge\alpha$ for all $a_1,a_2,...,a_n\in A$. A is called fuzzy bounded, if it is $\alpha-$bounded for each $\alpha\in(0,1)$.

Definition 6 [12]. Let X be a real normed linear space and let A be a nonempty, bounded subset of X. Let $F_A:X\to\mathbb{R}$ be the farthest distance function defined by $F_A(x)=\sup\{\|x-y\|:y\in A\}$. The set-valued map $Q:X\to A$ defined by $Q(x)=\{y'\in A:\|x-y'\|=F_A(x)\}$ is called the farthest-point map supported by A. Every element $y'\in Q(x)$

is called farthest point of A from x and the set of all farthest points of A is written as $\operatorname{far}(A)$. If $Q(x)$ is nonempty (respectively singleton) for each $x \in X$, then A is said to have the farthest-point property (respectively unique farthest point property).

Example 5. Let $\left(X=\mathbb{R}^2,\|\cdot\|\right)$ normed linear space and the set A is defined as $A=\{(x_1,x_2): -1 \le x_1 \le 1, -1 \le x_2 \le 1\}$.

Since $x=(2,2) \in X$, $F_A(2,2)=\sup\{\|(2,2)-(y_1,y_2)\|:(y_1,y_2)\in A\}=3\sqrt{2}$ therefore $Q(x)=\left\{y=(y_1,y_2)\in A: \|(2,2)-(y_1,y_2)\|=3\sqrt{2}\right\}$. Then $Q(x)=\{(-1,-1)\}$.

Since $x=(0,2)$, $Q(x)=\{(-1,-1),(1,-1)\}$.

Similarly, since $A=\{(x_1,x_2): -1 < x_1 \le 1, -1 \le x_2 \le 1\}$ and $x=(2,2)$, we obtain

$F_A(2,2)=\sup\{\|(2,2)-(y_1,y_2)\|:(y_1,y_2)\in A\}=3\sqrt{2}$ and $Q(x)=\left\{y=(y_1,y_2)\in A: \|(2,2)-(y_1,y_2)\|=3\sqrt{2}\right\}$.

However, because of $(-1,-1)\notin A$, $Q(x)=\varnothing$.

Example 6. Let $X=\mathbb{R}\times\mathbb{R}$, $x=(a,b)\in X$. We define a l_∞-norm on X as $\|(a,b)\|_\infty=\max\{|a|,|b|\}$ for $a,b\in\mathbb{R}$. If we take $A=\{(a_1,a_2): -4\le a_1 \le 2, a_2=0\}$ and $x=(3,0)$ then $F_A(x)=\sup\left\{\|(3,0)-(y_1,y_2)\|_\infty : (y_1,y_2)\in A\right\}=7$ and $Q(x)=\left\{y=(y_1,y_2)\in A: \|(3,0)-(y_1,y_2)\|=7\right\}$. So $Q(x)=\{(-4,0)\}$.

Since $A=\{(a_1,a_2): -6\le a_2\le 6, a_1=0\}$ and $x=(0,2)$, $F_A(x)=\sup\{\|(0,2)-(y_1,y_2)\|_\infty:(y_1,y_2)\in A\}=8$ and $Q(x)=\left\{y=(y_1,y_2)\in A: \|(0,2)-(y_1,y_2)\|=8\right\}$. So $Q(x)=\{(0,-6)\}$.

Similarly, If we take $A=\{(a_1,a_2): -3\le a_1\le 3, -1\le a_2\le 5\}$ and $x=(0,2)$ then $F_A(x)=3$ and $Q(x)=\{(y_1,5),(y_1,-5): -3\le y_1\le 3\}\cup\{(3,y_2),(-3,y_2): -5\le y_2\le 5\}$.

Definition 7 [15]. Let B be a bounded subset of a real normed linear space X and $x \in X$. An element $b_0\in B$ is called a strongly unique farthest point to X from B, if there exists a constant $K_x>0$ such that for every $b_0\in B$, $\|x-b_0\|\ge\|x-b\|+K_x\|b-b_0\|$.

Theorem 1 [15]. Let B be a compact subset in a normed linear space X. Then for every $x\in X$, there exists a farthest point from B.

Corollary 1 [15]. Let B be a closed and bounded finite dimensional set in a normed linear space X. Then for every $x \in X$, there exists a farthest point in B.

Corollary 2 [15]. Let f be a continuous mapping from a normed linear space X into another normed linear space Y. Let B be a compact set in X. Then for every $y \in Y$, there exists a farthest point from $f(B)$.

Definition 8 [11] . Let A be a fuzzy α – bounded subset of X. For $x_1,x_2,...,x_{n-1},x\in X$ and $0<s<t$, we define

$Q_\alpha(A,x)$ to be $\{a\in A : N(x_1,x_2,...,x_{n-1},x-a,s)\geq\alpha$ implies $N(x_1,x_2,...,x_{n-1},x-b,t)\geq\alpha$ for all $b\in A\}$.

If there is no danger of ambiguity, we denote $Q_\alpha(A,x)$ simply by $Q_\alpha(x)$.

Definition 9 [11]. Each a element of $Q_\alpha(x)$ is called a fuzzy $\alpha-$farthest point from x and the map $x\to Q_\alpha(x)$ is called the $\alpha-$farthest point map associated to A.

Definition 10 [11]. The set A is said to be fuzzy $\alpha-$remotal in X if for each $x\in X$, $Q_\alpha(x)$ is nonempty.

Definition 11 [11]. Let A be a fuzzy $\alpha-$bounded subset of X. If for each $a,b\in A$ and each $t>0$, the relation $N(x_1,x_2,...,x_{n-1},a-b,t)\geq\alpha$ is constant then A is called $\alpha-$singleton. A is singleton if and only if it is $\alpha-$singleton for all $\alpha\in(0,1)$.

Definition 12 [11]. If $Q_\alpha(x)$ is $\alpha-$singleton, then we say that x admits an $\alpha-$farthest point in X.

Definition 13 [11]. A is said to be fuzzy $\alpha-$uniquely remotal in X if each $x\in X$ admits $\alpha-$unique $\alpha-$farthest point in A.

Remark 2 [11]. If for some $x\in X$, $Q_\alpha(x)$ is not empty, then A is fuzzy $\alpha-$ bounded. To see this, let $a\in Q_\alpha(x)$. By (N5) there is an $s_0>0$ such that $N(x_1,x_2,...,x_{n-1},x-a,s_0)\geq\alpha$.

Moreover, we find some $t_1>0$ such that $N(x_1,x_2,...,x_{n-1},x,t_1)\geq\alpha$. Thus, for $m=t_0+t_1$ we have

$$N(x_1,x_2,...,x_{n-1},b,m)\geq\min N\{(x_1,x_2,...,x_{n-1},b-x,t_0),N(x_1,x_2,...,x_{n-1},x,t_1)\}\geq\alpha,$$

for each $b\in A$. This show that A is fuzzy $\alpha-$bounded. So, it assume that A is $\alpha-$bounded in our further discussion.

Lemma 1 [11]. Let A be a subset of X and Q_α be the $\alpha-$farthest point map related to A and $a\in A$. For some $q_\alpha(x)\in Q_\alpha(x)$ and all $t>0$

$$N(x_1,x_2,...,x_{n-1},a-q_\alpha(x),t)\geq\alpha, \tag{12}$$

then $a\in Q_\alpha(x)$.

Proof. Let $s<t$ and $N(x_1,...,x_{n-1},x-a,s)\geq\alpha$. Then for $\varepsilon=t-s$,

$$\begin{aligned} N\left(x_1,...,x_{n-1},x-q_\alpha(x),s+\frac{\varepsilon}{2}\right) &\geq\min\left\{N(x_1,...,x_{n-1},x-a,s),N\left(x_1,...,x_{n-1},a-q_\alpha(x),\frac{\varepsilon}{2}\right)\right\}\\ &\geq\min\{\alpha,\alpha\}\\ &=\alpha \end{aligned} \tag{13}$$

Let $b \in A$. Since $t = s + \varepsilon > s + \frac{\varepsilon}{2}$, the definition of $Q_\alpha(x)$ implies that $N(x_1, x_2, ..., x_{n-1}, x - b, t) \geq \alpha$. As a result $a \in Q_\alpha(x)$.

Corollary 2 [11]. Let A be a fuzzy $\alpha-$uniquely remotal subset of X. If $Q_\alpha(x) \cap Q_\alpha(y) \neq \varnothing$, then $Q_\alpha(x) = Q_\alpha(y)$.

Definition 14 [11]. Let A be a fuzzy bounded subset of X. For $x \in X$, we define $Q(A,x) = \bigcap_{\alpha \in (0,1)} Q_\alpha(A,x)$. For the sake of simplicity we call $Q(A,x)$ as $Q(x)$. Each element of $a \in Q(x)$ is called a fuzzy farhest point of A from x and the map $x \to Q(x)$ is called the fuzzy farthest point map associated to A.

Lemma 2 [11]. Let A be a fuzzy bounded subset of X. Then

i. $Q(x) = \{a \in A : \forall b \in A, N(x_1, x_2, ..., x_{n-1}, x - a, s) \leq N(x_1, x_2, ..., x_{n-1}, x - b, t) \text{ if } 0 < s < t\}$

ii. A is fuzzy uniquely remotal if and only if $Q(x)$ is singleton for each $x \in X$.

Theorem 2 [11]. Let A be a fuzzy $\alpha-$bounded subset of X and $x \to Q_\alpha(x)$ be the fuzzy $\alpha-$farthest point map. If $x_k \to_\alpha x$ and $q_\alpha(x_k) \in Q_\alpha(x_k)$ for each k and $q_\alpha(x_k) \to_\alpha y$, then $y \in Q_\alpha(x)$.

2. Main Results

In this section we will refer to comparing some results of the farthest point problems in classical $n-$ normed spaces to fuzzy normed spaces.

The following lemma show that our definitions in (X, N) fuzzy $n-$normed space are similar to definitions in classic $n-$normed space where $(X, \|\cdot, ..., \cdot\|)$ is n-normed linear space and

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} \dfrac{t}{t + \| x_1, x_2, ..., x_n \|} & ; \quad x_1, x_2, ..., x_n \text{ Linear independent} \\ 1 & ; \quad x_1, x_2, ..., x_n \text{ Linear dependent} \end{cases} \tag{14}$$

for all $x_1, x_2, ..., x_n \in X$ and $t > 0$.

Lemma 3. Let $(X, \|\cdot, ..., \cdot\|)$ be a $n-$ normed space and A be a fuzzy bounded subset of X. Then $\cap_{\alpha \in (0,1)} Q_\alpha(A,x)$ is equal to $Q(A,x)$ in $(X, \|\cdot, ..., \cdot\|)$.

Proof. Let take the $a_1, ..., a_{n-1}, x, a \in X$ and $a \in \cap_{\alpha \in (0,1)} Q_\alpha(A,x)$ arbitrary elements. Thus $N(x_1, x_2, ..., x_{n-1}, x - b, t) \geq N(x_1, x_2, ..., x_{n-1}, x - a, s)$ for all $b \in A$ and $0 < s < t$. If we use this inequality, for all $b \in A$,

$$\frac{t}{t+\|a_1,...,a_{n-1},x-b\|}=N(a_1,...,a_{n-1},x-b,t)\geq N(a_1,...,a_{n-1},x-a,s)=\frac{s}{s+\|a_1,...,a_{n-1},x-a\|}.$$

Hence for all $b\in A$ and $0<s<t$,

$$\begin{aligned}\frac{t}{t+\|a_1,...,a_{n-1},x-b\|} &\geq \frac{s}{s+\|a_1,...,a_{n-1},x-a\|}\\ \Rightarrow t.(s+\|a_1,...,a_{n-1},x-a\|) &\geq s.(t+\|a_1,...,a_{n-1},x-b\|)\\ \Rightarrow t.s+t.\|a_1,...,a_{n-1},x-a\| &\geq s.t+s.\|a_1,...,a_{n-1},x-b\|\\ \Rightarrow t.\|a_1,...,a_{n-1},x-a\| &\geq s.\|a_1,...,a_{n-1},x-b\|.\end{aligned}$$

If we write $s\to t$, we show that $\|a_1,...,a_{n-1},x-b\|\leq\|a_1,...,a_{n-1},x-a\|$. Since a is arbitrary, for all $a\in\cap_{\alpha\in(0,1)}Q_\alpha(A,x)$, $a\in Q(A,x)$ in $(X,\|\cdot,...,\cdot\|)$.

Conversely, We suppose that $a\in Q(A,x)$ is arbitrary element in $(X,\|\cdot,...,\cdot\|)$. Then for all $b\in A$ and $0<s<t$,

$$\|a_1,...,a_{n-1},x-b\|\leq\|a_1,...,a_{n-1},x-a\|$$

$$\Rightarrow t.\|a_1,...,a_{n-1},x-a\|\geq s.\|a_1,...,a_{n-1},x-a\|\geq s.\|a_1,...,a_{n-1},x-b\|$$

$$\Rightarrow s.t+t.\|a_1,...,a_{n-1},x-a\|\geq s.t+s.\|a_1,...,a_{n-1},x-b\|$$

$$\Rightarrow t(s+\|a_1,...,a_{n-1},x-a\|)\geq s(t+\|a_1,...,a_{n-1},x-b\|)$$

$$\Rightarrow\frac{t}{t+\|a_1,...,a_{n-1},x-b\|}\geq\frac{s}{s+\|a_1,...,a_{n-1},x-a\|}$$

$$\Rightarrow N(a_1,...,a_{n-1},x-b,t)\geq N(a_1,...,a_{n-1},x-a,s).$$

Since a is arbitrary, $a\in\cap_{\alpha\in(0,1)}Q_\alpha(A,x)$ for all $a\in Q(A,x)$.

Example 7. Let $\|\cdot,\cdot\|:X\times X\to\mathbb{R}$ be $2-$norm defined as $\|x_1,x_2\|=\left|\det\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}\right|$ where $X=\mathbb{R}\times\mathbb{R}$, $x_1=(x_{11},x_{12}),x_2=(x_{21},x_{22})\in X$. While $(X=\mathbb{R}^2,\|\cdot,\cdot\|)$ is 2-normed linear space, $A=\{(a_1,a_2):-1\leq a_1\leq 1,a_2=0\}$ and $x_1=(3,1)$, We calculate the farthest point map $x\to Q(A,x)$ for $x=(2,2)$ point.

$$\begin{aligned}F_A(x) &= \sup\{\|x_1, x-a\| : a \in A\} \\ &= \sup\{\|(3,1),(2,2)-(a_1,a_2)\| : (a_1,a_2) \in A\} \\ &= \sup\{\|(3,1),(2-a_1,2-a_2)\| : (a_1,a_2) \in A\} \\ &= \sup\{|3.(2-a_2)-1.(2-a_1)| : (a_1,a_2) \in A\} \\ &= \sup\{|4+a_1-3a_2| : (a_1,a_2) \in A\} \\ &= 4\end{aligned}$$

Then, the following set-valued map is obtained.

$$\begin{aligned}Q(A,x) &= \{a=(a_1,a_2) \in A : \|x_1, x-a\| = 5\} \\ &= \{(a_1,a_2) \in A : \|(3,1),(2,2)-(a_1,a_2)\| = 5\} \\ &= \{(a_1,a_2) \in A : \|(3,1),(2-a_1,2-a_2)\| = 5\} \\ &= \{(a_1,a_2) \in A : |3.(2-a_2)-1.(2-a_1)| = 5\} \\ &= \{(a_1,a_2) \in A : |4+a_1-3a_2| = 5\} \\ &= \{(1,0)\}\end{aligned}$$

In the following example, we examine the above example in the fuzzy $2-$ norm.

Example 8. Let $(X,\|\cdot,\cdot\|)$ be $2-$ normed linear space defined by Example 7 and $N : X \times X \times \mathbb{R} \to [0,1]$ be fuzzy $2-$ norm defined as

$$N(x,y,t) = \begin{cases} \dfrac{t}{t+\|x,y\|} & ; t>0, x,y \in X \\ 0 & ; t \le 0, x,y \in X. \end{cases}$$

While $A = \{(a_1,a_2) : -1 \le a_1 \le 1, a_2 = 0\}$ and $y=(3,1)$, We calculate the fuzzy farthest point map $x \to Q(A,x)$ for $x=(2,2)$ point.

For all $b \in A$ and $0<s<t$,

$$Q(A,x) = \{a=(a_1,a_2) \in A : N(y,x-a,s) \le N(y,x-b,t)\}.$$

Then

$$\begin{aligned}&N(y,x-a,s) \le N(y,x-b,t) \\ &\Rightarrow \frac{s}{s+\|y,x-a\|} \le \frac{t}{t+\|y,x-b\|} \\ &\Rightarrow s.\|y,x-b\| \le t.\|y,x-a\|.\end{aligned}$$

If we write $s \to t$, we show that $\|y, x-b\| \le \|y, x-a\|$.

For all $b \in A$, the $a \in A$ value providing this condition is the $(1,0) \in A$ value we calculated in example 7. So $Q(A,x) = \{(1,0)\}$.

As we can see from these two examples, the farthest point map and set made by the classical 2-norm and the fuzzy 2-norm is equal.

Let's examine a different example to see that when we change the fuzzy norm that we use, the same results can be achieved. Since this example is given in $3-$normed spaces, we firstly give $3-$normed space.

Example 9. Let $X = \mathbb{R}^3$ and $(X, \|\cdot,\cdot,\cdot\|_\infty)$ be a 3-normed linear space. Define $N : X \times X \times X \times \mathbb{R} \to [0,1]$, by

$$N(x_1, x_2, x_3, t) = \begin{cases} \left(e^{\frac{\|x_1, x_2, x_3\|_\infty}{t}} \right)^{-1} & ; \quad t > 0, x_1, x_2, x_3 \in X \\ 0 & ; \quad t \le 0, x_1, x_2, x_3 \in X \end{cases}$$

for all $x_1, x_2, x_3 \in X$ and $t \in \mathbb{R}$, where $\|x_1, x_2, x_3\|_\infty = \max\limits_{1 \le i \le 3} \sum\limits_{j=1}^{3} |x_{ij}|$.

Then N is fuzzy 3-norm and (X, N) is fuzzy 3-normed linear space.

Example 10. Let $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $x_i = (x_{i1}, x_{i2}, x_{i3}) \in X$ for $i = 1,2,3$. Define

$\|\cdot,\cdot,\cdot\| : X \times X \times X \to \mathbb{R}$ by $\|x_1, x_2, x_3\|_\infty = \max\limits_{1 \le i \le 3} \sum\limits_{j=1}^{3} |x_{ij}|$. So $(X = \mathbb{R}^3, \|\cdot,\cdot,\cdot\|)$ is 3-normed linear space.

While $A = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 + a_2^2 \le 1, 0 \le a_3 \le a_1^2 + a_2^2\}$, $x_1 = (1,0,0)$ and $x_2 = (0,1,0)$, we calculate the farthest point map $x \to Q(A,x)$ for $x = (0,0,4)$ point.

$$\begin{aligned} F_A(x) &= \sup\{\|x_1, x_2, x - a\|_\infty : a \in A\} \\ &= \sup\{\|(1,0,0), (0,1,0), (0,0,4) - (a_1, a_2, a_3)\|_\infty : (a_1, a_2, a_3) \in A\} \\ &= \sup\{\|(1,0,0), (0,1,0), (-a_1, -a_2, 4 - a_3)\|_\infty : (a_1, a_2, a_3) \in A\} \\ &= \sup\{\max\{1, 1, |a_1| + |a_2| + |4 - a_3|\} : (a_1, a_2, a_3) \in A\} \\ &= 4 + \sqrt{2} \end{aligned}$$

Then, the following set-valued map is obtained.

$$\begin{aligned} Q(A,x) &= \left\{a=(a_1,a_2,a_3)\in A: \|x_1,x_2,x-a\|_\infty = 4+\sqrt{2}\right\} \\ &= \left\{(a_1,a_2,a_3)\in A: \|(1,0,0),(0,1,0),(0,0,4)-(a_1,a_2,a_3)\|_\infty = 4+\sqrt{2}\right\} \\ &= \left\{(a_1,a_2,a_3)\in A: \|(1,0,0),(0,1,0),-(a_1,a_2,a_3-4)\|_\infty = 4+\sqrt{2}\right\} \\ &= \left\{(a_1,a_2,a_3)\in A: \max\{1,1,|a_1|+|a_2|+|a_3-4|\} = 4+\sqrt{2}\right\} \\ &= \left\{(a_1,a_2,a_3)\in \mathbb{R}^3: |a_1|+|a_2|+|a_3-4| = 4+\sqrt{2}, a_1^2+a_2^2\le 1, 0\le a_3\le a_1^2+a_2^2\right\}. \end{aligned}$$

Thus,

$$Q(A,x)=\left\{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\left(\frac{\sqrt{2}}{2},\frac{-\sqrt{2}}{2},0\right),\left(\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2},0\right)\right\}.$$

Example 11. Let $X=\mathbb{R}^3$ and $(X,\|\cdot,\cdot,\cdot\|_\infty)$ be 3-normed linear space. Define $N: X\times X\times X\times\mathbb{R}\to[0,1]$ fuzzy 3-norm by

$$N(x_1,x_2,x_3,t)=\begin{cases}\left(e^{\frac{\|x_1,x_2,x_3\|_\infty}{t}}\right)^{-1} & ; \quad t>0 \\ 0 & ; \quad t\le 0\end{cases}$$

for all $x_1,x_2,x_3\in X$ and $t\in\mathbb{R}$ where $\|x_1,x_2,x_3\|_\infty = \max\limits_{1\le i\le 3}\sum_{j=1}^{3}|x_{ij}|$. So (X,N) is fuzzy 3-normed space.

While $A=\left\{(a_1,a_2,a_3)\in\mathbb{R}^3: a_1^2+a_2^2\le 1, 0\le a_3\le a_1^2+a_2^2\right\}$, $x_1=(1,0,0)$ and $x_2=(0,1,0)$, we calculate the fuzzy farthest point map $x\to Q(A,x)$ for $x=(0,0,4)$ point.

For all $b\in A$ and $0<s<t$,

$$Q(A,x)=\left\{a=(a_1,a_2,a_3)\in A: N(x_1,x_2,x-a,s)\le N(x_1,x_2,x-b,t)\right\}.$$

Then,

$$\begin{aligned} & N(x_1,x_2,x-a,s)\le N(x_1,x_2,x-b,t) \\ &\Rightarrow \left(e^{\frac{\|x_1,x_2,x-a\|_\infty}{s}}\right)^{-1} \le \left(e^{\frac{\|x_1,x_2,x-b\|_\infty}{t}}\right)^{-1} \\ &\Rightarrow \frac{\|x_1,x_2,x-a\|_\infty}{s} \ge \frac{\|x_1,x_2,x-b\|_\infty}{t} \\ &\Rightarrow t.\|x_1,x_2,x-a\|_\infty \ge s.\|x_1,x_2,x-b\|_\infty \end{aligned}$$

If we write $s\to t$, we show that

$$\|x_1, x_2, x-a\|_\infty \geq \|x_1, x_2, x-b\|_\infty .$$

For all $b \in A$, the $a \in A$ value providing this condition is the $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\left(\frac{\sqrt{2}}{2},\frac{-\sqrt{2}}{2},0\right),\left(\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2},0\right) \in A$ value we calculated in example 10. So $Q(A,x)=\left\{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right),\left(\frac{\sqrt{2}}{2},\frac{-\sqrt{2}}{2},0\right),\left(\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2},0\right)\right\}$.

As we can see from Example 10 and Example 11, the farthest point map and set made by the classical 3-norm and the fuzzy 3-norm is equal.

3. Conclusion

The farthest point map and set varies with the given norm. But the farthest point map and set calculated in fuzzy n-normed space created using the norm given in classical space don't varies. As a result, as long as the norm we use in classical space doesn't change, calculated farthest point map and set is equal in classic and fuzzy n-normed spaces.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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