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Statistical Hypo-Convergence in Sequences of Functions

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Abstract

In this paper, we define statistical hypo-convergence in metric spaces as an alternative to statistical pointwise and uniform statistical convergence. We show that this type of convergence provides a useful tool for solving stochastic optimization and variational problems. Also, its characterizations with level sets are obtained.

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Keywords

Statistical hypo-convergence, hypographs, upper semicontinuity, level sets.

1. Introduction.

Hypo-convergence focuses on hypo-graphs whereas epi-convergence deals with epigraphs. In literature, epi-convergence is more familiar than hypo-convergence and it is first studied by Wijsman [20, 21] where it is called infimal convergence in the late of 1960's. After Wijsman's initial contributions, it is studied by Mosco [12] on variational inequalities, by Joly [8] on topological structures compatible with epi-convergence, by Salinetti and Wets [17] on equisemicontinuous families of convex functions, by Attouch [2] on the relationship between the epi-convergence of convex functions and the graphical convergence of their subgradient mappings, and by McLinden and Bergstrom [11] on the preservation of epi-convergence under various operations performed on convex functions. Furthermore, Dal Maso [10] called it Γ-convergence. The term epi-convergence is used by Wets [19] in 1980 for the first time. Epi-convergence is needed to solve some mathematical problems including stochastic optimization, variational problems and partial differential equations.

In this part fundamental definitions and theorems will be given. First of all, let (X, d) be a metric space and f, (f_n) are functions defined on X with $n \in \mathbb{N}$. If it is not mentioned explicitly the symbol d stands for the metric on X.

Let $K \subseteq \mathbb{N}$ and if the limit $\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|$ exists then it is called asymptotic density of K where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n (see[1, 13]).

$$\text{If } \delta(K_1) = \delta(K_2) = 1, \text{ then } \delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 1.$$
$$\text{If } \delta(K_1) = \delta(K_2) = 0, \text{ then } \delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 0.$$

Statistical convergence of a sequence of scalars was introduced by Fast [5]. Let $x = (x_k)$ be a sequence of real or complex numbers. If for all $\varepsilon > 0$, there exists L such that,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

then the sequence (x_k) is statistically convergent to L.

The concepts of statistical limit superior and statistical limit inferior were introduced by Fridy and Orhan [6]. Let k be a positive integer and x be a real number sequence. Define the sets B_x and A_x as

$$B_x = \{b \in \mathbb{R} : \delta(\{n : x_n > b\}) \neq 0\}, \quad A_x = \{a \in \mathbb{R} : \delta(\{n : x_n < a\}) \neq 0\}.$$

Then statistical limit superior and statistical limit inferior of x is given by

$$st\text{-}\limsup x = \begin{cases} \sup B_x & if \quad B_x \neq \emptyset, \\ -\infty & if \quad B_x = \emptyset. \end{cases}$$

$$st\text{-}\liminf x = \begin{cases} \inf A_x & if \quad A_x \neq \emptyset, \\ +\infty & if \quad A_x = \emptyset. \end{cases}$$

Lemma 1.1 [6] *If* β = st- lim sup x *is finite, then for every* $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0 \ and \ \delta(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0 \tag{1.1}$$

Conversely, if (1.1) holds for every, $\varepsilon > 0$ then β = st- lim sup x.

The dual statement for st- lim inf x is as follows:

Lemma 1.2 [6] *If* α = st- lim inf x *is finite, then for every* $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}) \neq 0 \ and \ \delta(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}) = 0 \tag{1.2}$$

Conversely, if (1.2) holds for every, $\varepsilon > 0$ *then* α = st- lim sup x.

A point $\xi \in X$ is called a statistical limit point of a sequence $x = (x_k)$ if there is a set $K = k_1 < k_2 < k_3 < \ldots$ with $\delta(K) \neq 0$ such that $x_{k_n} \to \xi$ as $n \to \infty$. The set of all statistical limit points of a sequence x will be denoted by Λ_x.

A point $\xi \in X$ is called a statistical cluster point of $x = (x_k)$ if for any $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon\}) \neq 0.$$

The set of all statistical cluster points of x will be denoted by Γ_x.

Let L_x denote the set of all limit points ξ (accumulation points) of the sequence x; i.e. $\xi \in L_x$ if there exists an infinite set $K = k_1 < k_2 < k_3 < \ldots$ such that $x_{k_n} \to \xi$ as $n \to \infty$.

Obviously we have $\Lambda_x \subseteq \Gamma_x \subseteq L_x$.

In our study we will be interested much more on sequence of functions. Statistical convergence on sequence of functions is defined by Gokhan and Gungor [7].

Following definitions are statistical inner and outer limits on the concept of set convergence which is fundamental to define statistical hypo-limit by using sets. In this paper, we deal with Painleve-Kuratowski [9] convergence and actually its statistical version will be studied here which is defined by Sever and Talo [18]. In set convergence, following collections of subsets of $\mathbb{N}$ play an important role for defining statistical inner and outer limits on sequence of sets.

$$\mathcal{S} = \{N \subset \mathbb{N} : \delta(N) = 1\},$$
$$\mathcal{S}^{\#} = \{N \subset \mathbb{N} : \delta(N) \neq 0\}.$$

Definition 1.3 [18] *Let (X, d) be a metric space. Statistical inner and outer limit of a sequence (A_n) of closed subsets of X are defined as follows*:

$$st\text{-}\liminf_n A_n = \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}, \forall n \in N : A_n \cap V \neq \emptyset\}, \tag{1.3}$$

$$st\text{-}\liminf_n A_n = \{x \mid \forall \varepsilon > 0, \ \exists N \in \mathcal{S}, \ \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset\} \tag{1.4}$$

$$st\text{-}\limsup_n A_n = \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}^{\#}, \forall n \in N : A_n \cap V \neq \emptyset\} \tag{1.5}$$

$$st\text{-}\limsup_n A_n = \{x \mid \forall \varepsilon > 0, \ \exists N \in \mathcal{S}^{\#}, \ \forall n \in N : A_n \cap B(x, \varepsilon) \neq \emptyset\} \tag{1.6}$$

Proposition 1.4 [18] *Let (X, d) be a metric space and (An) be a sequence of closed subsets of X.*

Then

$$st\text{-}\liminf_n A_n = \{x \mid \exists N \in \mathcal{S}, \forall n \in N, \exists y_n \in A_n : \lim_n y_n = x\}.$$

Proposition 1.5 [18] *Let (X, d) be a metric space and (An) be a sequence of closed subsets of X.*
Then

$$\mathrm{st}-\limsup_n A_n = \{x | \exists N \in S^{\#}, \forall n \in N, \exists y_n \in A_n : x \in \Gamma_{y_n}\}$$

Let f be a function defined on *X*, the hypograph of f is the set hypo(*f*) = $\{(x, \alpha) \in X \times \mathbb{R} | \alpha \leq f(x)\}$ and its level set is defined by $lev_{\geq\alpha} f = \{x \in X \mid f(x) \geq \alpha\}$.

Definition 1.6 [15] *For any sequence (fn) of functions on X, the lower hypo-limit, h-* lim $\inf_n f_n$*, is the function having as its hypograph the outer limit of the sequence of sets hypo* f_n:

$$hypo(h\text{-}\liminf_n f_n) = \liminf_n(hypo f_n).$$

The upper hypo-limit, h- lim $\sup_n f_n$*, is the function having as its hypograph the inner limit of the sequence of sets hypo*f_n*:*

$$hypo(h\text{-}\limsup_n f_n) = \limsup_n(hypo f_n).$$

When these two functions equal to each other, we have h- $\lim_n f_n$ = *h-* lim $\inf_n f_n$ = *h-* lim $\sup_n f_n$. *Hence the functions* f_n *are said to hypo-convergent to the function f. It is symbolized by* $f_n \xrightarrow{h} f$ *Moreover, the relation between set convergence and convergence of sequence of functions appears in the following equality.*

$$f_n \xrightarrow{h} f \Leftrightarrow hypo f_n \to hypo(f).$$

We also advise to look at [3, 4, 14, 16] for detailed information about new types of convergence of sequences of real valued functions and statistical convergence.

2. Main Result

In this part, statistical hypo-convergence is defined by the help of Kuratowski convergence on sets. The functions will be taken upper semicontinuous in order to use properties on closed sets since hypo-graphs of upper semicontinuous functions are closed. Set properties will give a new characterization of statistical hypo-convergence by using neighbourhoods of the point $x \in X$ in a metric space. Neighbourhoods will give other characterizations of statistical hypo-convergence by using sequences this time. Level sets which are important instruments in set theory are also included in our calculations for lower and upper statistical hypo-limits. Moreover, statistical hypo-convergence and statistical upper semicontinuity will be discussed at the end.

Definition 2.1 *Let (X, d) be a metric space and* (f_n) *a sequence of upper semicontinuous functions defined from X to* $\overline{\mathbb{R}}$. *The lower statistical hypo-limit,* h_{st}- lim $\inf_n f_n$ *is defined by the help of the sequence of sets*:

$$hypo(h_{st}\text{-}\liminf_n f_n) = st\text{-}\liminf_n(hypo f_n).$$

Similarly, the upper statistical hypo-limit h_{st}- lim $\sup_n f_n$ *is defined:*

$$hypo(h_{st}\text{-}\limsup_n f_n) = st\text{-}\limsup_n(hypo f_n).$$

When these two functions are equal, we get statistical hypo-limit function:

$$f = h_{st}\text{-}\lim_n f_n = h_{st}\text{-}\limsup_n f_n = h_{st}\text{-}\liminf_n f_n.$$

As defined in above, it is obvious that h_{st}- lim $\inf_n f_n \leq h_{st}$- lim $\sup_n f_n$.

Here we use statistical Painleve-Kuratowski convergence. Whenever (f_n) is hypo-convergent to f we can use the inclusion $st\text{-}\limsup_n(hypof_n) \subset hypof \subset st\text{-}\liminf_n(hypof_n)$.

Moreover, following comparisons with hypo-limits are valid for every function $f : X \to \overline{\mathbb{R}}$.

$$h\text{-}\liminf_n f_n \leq h_{st}\text{-}\liminf_n f_n \ , \quad h\text{-}\limsup_n f_n \leq h_{st}\text{-}\limsup_n f_n.$$

In the following example, the function is not hypo-convergent whereas it has statistical hypo-limit.

Example 2.2 *Given a sequence* $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ *defined as*

$$f_n(x) = \begin{cases} 2nxe^{-2n^2x^2} & if\ n\ is\ square, \\ nxe^{-2n^2x^2} & if\ n\ is\ nonsquare. \end{cases}$$

$$h\text{-}\limsup_n f_n(x) = \begin{cases} e^{-\frac{1}{2}} & if \quad x = 0, \\ 0 & if \quad x \neq 0, \end{cases}$$

$$h\text{-}\liminf_n f_n(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}} & if \quad x = 0, \\ 0 & if \quad x \neq 0, \end{cases}$$

$$h_{st}\text{-}\lim_n f_n(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}} & if \quad x = 0, \\ 0 & if \quad x \neq 0. \end{cases}$$

In this example, statistical hypo-convergence deals with supremum values of a sequence of functions even if ordinary hypo-convergence fails to find it. Hence, it makes statistical hypo-convergence more reliable for finding optimum values in stochastic optimization problems.

On the other hand, (f_n) is statistically pointwise convergent to the function $f(x) = 0$ which is different than hypo-limits. It shows us that statistical hypo-convergence is neither stronger nor weaker than statistical pointwise convergence. The obvious difference between these convergence types is obtaining supremums.

Now we will give characterizations of upper and lower statistical hypo-convergence using neighbourhoods. Before giving definitions, we need to prove following Lemma 2.3 and Lemma 2.4.

Lemma 2.3 *Let* (X, d) *be a metric space and* (f_n) *a sequence of upper semicontinuous functions defined from* X *to* $\overline{\mathbb{R}}$*, for every* $x \in X$*, define* $g : X \to \overline{\mathbb{R}}$ *by*

$$g(x) = \inf_{V \in \mathcal{N}(x)} st\text{-}\limsup_n \sup_{y \in V} f_n(y).$$

Then $st\text{-}\limsup_n(hypof_n) = hypo(g)$.

Proof: We should establish the epigraphical inclusions of the sets $st\text{-}\limsup_n(hypof_n) \subset hypo(g)$ and $hypo(g) \subset st\text{-}\limsup_n(hypof_n)$. For the first inclusion, let $(x, \alpha) \in st\text{-}\limsup_n(hypof_n)$ be arbitrary. Let $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$ be fixed. By definition of the statistical upper limit of sets, $\exists N \in \mathcal{S}^\#$ such that $\forall n \in N$ we have

$$V_0 \times (\infty, \alpha - \varepsilon) \bigcap hypof_n \neq \emptyset.$$

As a result,

$$\delta(\{n \in \mathbb{N} : \sup_{y \in V_0} f_n(y) > \alpha - \varepsilon\}) \neq 0$$

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By Lemma 1.1 we have,

$$st\text{-}\limsup_n \sup_{y\in V_0} f_n(y) \geq \alpha.$$

V_0 and ε were arbitrary, we have $g(x) > \alpha$ and hence $(x,\alpha) \in hypo(g)$.

For the second inclusion let $(x,\alpha) \in hypo(g)$, for all $V_0 \in \mathcal{N}(x)$ and for all $\varepsilon > 0$ we have,

$$\alpha \leq g(x) \leq st\text{-}\limsup_n \sup_{y\in V_0} f_n(y).$$

Again by Lemma 1.1, we get $\delta(\{n \in \mathbb{N}: \sup_{y\in V_0} f_n(y) > \alpha - \varepsilon\}) \neq 0$. It means, $\exists N \in \mathcal{S}^\#$ such that $\forall n \in N$

$$V_0 \times (\infty, \alpha - \varepsilon) \bigcap hypo f_n \neq \emptyset.$$

Hence $(x, \alpha) \in st\text{-}\limsup_n (hypo f_n)$.

Lemma 2.4 *Let* (X, d) *be a metric space and* (f_n) *a sequence of upper semicontinuous functions defined from* X *to* $\overline{\mathbb{R}}$, *for every* $x \in X$, *define* $h : X \to \overline{\mathbb{R}}$ *by*

$$h(x) = \inf_{V\in\mathcal{N}(x)} st\text{-}\liminf_n \sup_{y\in V} f_n(y).$$

Then $st\text{-}\liminf_n (hypo f_n) = hypo(h)$

Proof: We want to show $st\text{-}\limsup_n (hypo f_n) \subset hypo(h)$ and $hypo(h) \subset st\text{-}\liminf_n (hypo f_n)$. For the first inclusion, let $(x,\alpha) \in st\text{-}\limsup_n (hypo f_n)$ be arbitrary. Let $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$ be fixed. By definition of the statistical lower limit of sets, $\exists N \in \mathcal{S}$ such that $\forall n \in N$ we have

$$V_0 \times (\infty, \alpha - \varepsilon) \bigcap hypo f_n \neq \emptyset.$$

As a result,

$$\delta(\{n \in \mathbb{N} : \sup_{y\in V_0} f_n(y) < \alpha - \varepsilon\}) = 0$$

By Lemma 1.1 we have,

$$\text{st-}\liminf_n \sup_{y\in V_0} f_n(y) \geq \alpha.$$

V_0 and ε was arbitrary, we have $h(x) \geq \alpha$ and hence $(x,\alpha) \in hypo(h)$.
For the second inclusion, fix $(x,\alpha) \in hypo(h)$. Given $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$, $\exists N \in \mathcal{S}$ such that $\forall n \in N$ we have

$$\text{st-}\liminf_n \sup_{y\in V_0} f_n(y) \geq h(x) > \alpha - \varepsilon$$

and it equals to the following equality

$$\delta(\{n \in \mathbb{N} : \sup_{y\in V_0} f_n(y) > \alpha - \varepsilon\}) = 1.$$

Hence,

$$\delta(\{n \in \mathbb{N} : V_0 \times (\infty, \alpha - \varepsilon) \bigcap hypo f_n \neq \emptyset\}) = 1.$$

We conclude that

$$\delta(\{n \in \mathbb{N} : V_0 \times (\alpha-\varepsilon, \alpha+\varepsilon) \bigcap hypo f_n \neq \emptyset\}) = 1.$$

It gives $(x, \alpha) \in st\text{-}\liminf_n(hypof_n)$ and concludes the proof.

Next definition gives us a characterization of hypo-limits with the help of Lemma 2.3 and Lemma 2.4.

Definition 2.5 *Let (X, d) be a metric space and (f_n) a sequence of lower semicontinuous functions from X into $\overline{\mathbb{R}}$, for every $x \in X$, lower and upper statistical epi-limit functions are defined by*

$$\left(h_{st}\text{-}\liminf_n f_n\right)(x) = \inf_{V \in \mathcal{N}(x)} st\text{-}\liminf_n \sup_{y \in V} f_n(y)$$

$$\left(h_{st}\text{-}\limsup_n f_n\right)(x) = \inf_{V \in \mathcal{N}(x)} st\text{-}\limsup_n \sup_{y \in V} f_n(y)$$

If there exists a function $f : X \to \overline{\mathbb{R}}$ such that $h_{st}\text{-}\liminf_n f_n = h_{st}\text{-}\limsup_n f_n = f$, then we write $f = h_{st}\text{-}\lim_n f_n$ and we say that (f_n) is h_{st}-convergent to f on X.

Lemma 2.6 *Let $x = (x_n)$ be a real sequence. Then*

$$st\text{-}\liminf_{n\to\infty} x_n = \inf_{N \in \mathcal{S}\#} \sup_{n \in N} x_n = \sup_{N \in \mathcal{S}} \inf_{n \in N} x_n$$

$$st\text{-}\limsup_{n\to\infty} x_n = \sup_{N \in \mathcal{S}\#} \inf_{n \in N} x_n = \inf_{N \in \mathcal{S}} \sup_{n \in N} x_n$$

By lemma 2.6, the statistical hypo-limit infimum can be expressed as follows:

$$(h_{st}\text{-}\liminf_n f_n)(x) = \inf_{V \in \mathcal{N}(x)} \inf_{N \in \mathcal{S}\#} \sup_{n \in N} \sup_{y \in V} f_n(y) = \inf_{V \in \mathcal{N}(x)} \sup_{N \in \mathcal{S}} \inf_{n \in N} \sup_{y \in V} f_n(y).$$

Similarly, the statistical hypo-limit supremum can be expressed as follows:

$$(h_{st}\text{-}\limsup_n f_n)(x) = \inf_{V \in \mathcal{N}(x)} \sup_{N \in \mathcal{S}\#} \inf_{n \in N} \sup_{y \in V} f_n(y) = \inf_{V \in \mathcal{N}(x)} \inf_{N \in \mathcal{S}} \sup_{n \in N} \sup_{y \in V} f_n(y)$$

Proposition 2.7 *In a metric space (X, d) for every $x \in X$, the following inequalities hold:*

$$(h_{st}\text{-}\liminf_n f_n)(x) \geq st\text{-}\liminf_n f_n(x) \;, \qquad (h_{st}\text{-}\limsup_n f_n)(x) \geq st\text{-}\limsup_n f_n(x).$$

Proof: $\forall x \in X$, $\forall V \in \mathcal{N}(x)$, $\exists N \in \mathcal{S}$ such that $\forall n \in N$ we have

$$\sup_{y \in V} f_n(y) \geq f_n(x) \;, \qquad \sup_{y \in V} f_n(y) \geq f_n(x).$$

Since by the choice of our index set ($n \in N$), we get the following inequalities,

$$st\text{-}\liminf_n \sup_{y \in V} f_n(y) \geq st\text{-}\liminf_n f_n(x) \;, \qquad st\text{-}\limsup_n \sup_{y \in V} f_n(y) \geq st\text{-}\limsup_n f_n(x).$$

After taking the infimum over all $V \in \mathcal{N}(x)$ we get the desired conclusion.

Theorem 2.8 *Let (X, d) be a metric space and let (f_n) be a sequence of upper semicontinuous functions. Suppose that for each $\alpha \in \mathbb{R}$, $\exists(\alpha_n)$ of reals statistically convergent to α with $lev_{\geq\alpha} f = st\text{-}\lim_n(lev_{\geq\alpha_n} f_n)$, then $f = h_{st}\text{-}\lim_n f_n$.*

Proof: The condition $lev_{\geq\alpha}f \subset st\text{-}\liminf_n(lev_{\geq\alpha_n}f_n)$ valid for each $\alpha\in\mathbb{R}$ and for some sequence $\alpha_n \overset{st}{\to} \alpha$. Let $(x,\alpha)\in hypo(f)$ then there exists a sequence α_n statistically convergent to α such that $lev_{\geq\alpha}f \subset st\text{-}\liminf_n(lev_{\geq\alpha_n}f_n)$. Hence $x \in st\text{-}\liminf_n(lev_{\geq\alpha_n}f_n)$. It means there exists a sequence (x_n) statistically convergent to x such that $x_n \in (lev_{\geq\alpha_n}f_n)$. Finally we get $(x_n, \alpha_n) \overset{st}{\to} (x,\alpha)$ and $(x,\alpha)\in st\text{-}\liminf_n hypof_n$.

In order to get $st\text{-}\limsup_n hypof_n \subset hypo(f)$, suppose to the contrary that $(x, \beta) \in st\text{-}\limsup_n hypof_n$ but that $(x,\beta)\notin hypo(f)$. Then $\beta > f(x)$. We can find $N\in\mathcal{S}^{\#}$ such that $\forall n\in N$ $(x_n, \beta_n)\in hypof_n$ such that $(x, \beta) \in \Gamma(x_n, \beta_n)$. Choose a scalar α between β and $f(x)$ and let (α_n) be a sequence statistically convergent to α for which $lev_{\geq\alpha}f \supset st\text{-}\limsup_n(lev_{\geq\alpha_n}f_n)$. We have $\delta(n{:}\,\beta_n > \alpha_n) \neq 0$ and $(x_n, \beta_n)\in hypof_n$. $\exists N\in\mathcal{S}^{\#}$, $\forall n\in N$, $x_n \in lev_{\geq\alpha_n}f_n$ which means $x \in st\text{-}\limsup lev_{\geq\alpha_n}f_n$. By the inclusion $st\text{-}\limsup_n lev_{\geq\alpha_n}f_n \subset lev_{\geq\alpha}f$, we get $x \in lev_{\geq\alpha}f$ and $f(x) \geq \alpha$ which is a contradiction.

Theorem 2.9 *Let (f_n) and f be functions from X to $\overline{\mathbb{R}}$ with f upper semicontinuous. $hypo(f) \subset st\text{-}\liminf_n hypof_n$ if and only if for every open set $\mathcal{O}$ with $hypo(f)\cap\mathcal{O}\neq\emptyset$; there exists $N\in\mathcal{S}$ such that $hypof_n\cap\mathcal{O}\neq\emptyset$ for all $n\in N$.*

Proof: Necessity comes directly from (1.3). To illustrate sufficiency, suppose that there exists x in $hypo(f)$ but not in $st\text{-}\liminf_n hypof_n$. But then by (1.3), there exists an open neighborhood V of x such that for every $N\in\mathcal{S}$ there exists $n\in N$ with $V\cap hypof_n = \emptyset$; and also $V\cap hypo(f)\neq\emptyset$. This is the negation of the condition on the right.

Theorem 2.10 *Let (f_n) and f be functions from X to $\overline{\mathbb{R}}$ with f upper semicontinuous. $hypo(f) \supset st\text{-}\limsup_n hypof_n$ if and only if for every compact set C with $hypo(f)\cap C = \emptyset$ there exists $N\in\mathcal{S}$ such that $hypof_n\cap C = \emptyset$ for all $n\in N$.*

Proof: Let $hypo(f) \supset st\text{-}\limsup_n hypof_n$ and let there exists a compact set C with $hypo(f)\cap C = \emptyset$, such that for any $N\in\mathcal{S}$ one has $hypof_n\cap C\neq\emptyset$ for some $n\in N$. But then there exists $N\in\mathcal{S}^{\#}$ and a statistically convergent sequence $x_n \in hypof_n$ for $n\in N$ whose statistical limit not in $hypo(f)$, this is a contradiction. On the other hand, if there exists x in $st\text{-}\limsup_n hypof_n$ which is not in $hypo(f)$ then from (1.6), a ball $B(x, \varepsilon)$ with sufficiently small radius ε does not meet $hypo(f)$ yet meets $hypof_n$ for infinitely many n; this contradicts the condition on the right.

Corollary 2.11 *Let (f_n) and f be functions from X to $\overline{\mathbb{R}}$ with f upper semicontinuous. $hypo(f) \subset st\text{-}\liminf_n hypof_n$ if and only if whenever $hypo(f)\cap B(x,\varepsilon)\neq\emptyset$ for a ball $B(x,\varepsilon)$, there exists $N\in\mathcal{S}$ such that $hypof_n\cap B(x,\varepsilon)\neq\emptyset$ for all $n\in N$.*

Proof: It is clear from Theorem 2.9.

Corollary 2.12 *Let (f_n) and f be functions from X to $\overline{\mathbb{R}}$ with f upper semicontinuous. $hypo(f) \subset st\text{-}\limsup_n hypof_n$ if and only if whenever $hypo(f)\cap \overline{B}(x,\varepsilon)=\emptyset$ for a ball $\overline{B}(x,\varepsilon)$, there exists $N\in\mathcal{S}$ such that $hypof_n\cap\overline{B}(x,\varepsilon)=\emptyset$ for all $n\in N$.*

Proof: It is clear from Theorem 2.10.

Theorem 2.13 *Let (f_n) and f be functions from X to $\overline{\mathbb{R}}$ with f upper semicontinuous. $h_{st}\text{-}\liminf_n f_n \leq f$ if and only if $st\text{-}\limsup_n(\sup_C f_n) \leq \sup_C f$ for every compact set $C\subset X$.*

Proof: For necessity, assume that $h_{st}\text{-}\liminf_n f_n \leq f$ which means $st\text{-}\limsup_n(hypof_n)\subset hypo(f)$. Let us take any compact set $C\subset X$ and a value $\alpha\in\overline{\mathbb{R}}$ such that $\sup_C f<\alpha$. Then the compact set (C, α) in $X\times\overline{\mathbb{R}}$ does not meet $hypo(f)$. Hence by Theorem 2.10 there exists $N\in\mathcal{S}$ such that for all $n\in N$, (C, α) does not meet $hypof_n$ either. This implies $\sup_C f_n \leq \alpha$ and because of the choice of our index set we obtain $st\text{-}\limsup_n(\sup_C f_n) \leq \sup_C f$.

For sufficiency, let $B^+((x, \alpha), \varepsilon)$ be a cylinder does not meet $hypo(f)$ defined as,

$$B^+((x,\alpha),\varepsilon) := \overline{B}(x,\varepsilon)\times[\alpha-\varepsilon,\alpha+\varepsilon].$$

Given f is upper semicontinuous, so that $hypo(f)$ is closed and $\sup_{\overline{B}(x,\varepsilon)} f < \alpha-\varepsilon$. Since the ball

$\overline{B}(x,\varepsilon)$ is a compact set in X, we know by assumption that st- lim $\inf_n$ $(\sup_{\overline{B}(x,\varepsilon)} f) \leq \alpha - \varepsilon$. Therefore there exists $N \in \mathcal{S}$ such that $\sup_{\overline{B}(x,\varepsilon)} f_n < \alpha - \varepsilon$ for all $n \in N$. Then the cylinder $B^+((x, \alpha), \varepsilon)$ does not meet $hypof_n$ for any $n \in N$. By Corollary 2.12 we have st- lim $\sup_n$ $(hypof_n) \subset hypo(f)$ and we obtain h_{st}- lim $\inf_n f_n \leq f$.

Theorem 2.14 *Let* (f_n) *and* f *be functions from* X *to* $\overline{\mathbb{R}}$ *with* f *upper semicontinuous.* h_{st}- lim $\sup_n f_n \geq f$ *if and only if* $st\text{-}\lim\inf_n(\sup_{\mathcal{O}} f_n) \geq \sup_{\mathcal{O}} f$ *for every open set* $\mathcal{O} \subset X$.

Proof: The proof is similar to the previous one.

Definition 2.15 *The sequence* (f_n) *is called statistically equi upper semicontinuous at a point* x *if and only if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *and* $N \subset \mathcal{S}$ *such that for all* $y \in B(x, \delta)$ *we have,*

$$f_n(x) + \varepsilon > f_n(y)$$

for each $n \in N$.

Next theorem gives the basic condition that statistical convergence and statistical hypo-convergence coincide.

Theorem 2.16 (f_n) *and* f *are functions from* X *to* $\mathbb{R}$, *let* (f_n) *be statistically equi upper semi-continuous at* x. (f_n) *is statistically hypo-convergent to* f *at* x *if and only if* (f_n) *is statistically convergent to* f *at* x.

Proof: Assuming (f_n) is statistically upper semicontinuous at x, we have that for all $\varepsilon > 0$, there exists $V \in \mathcal{N}(x)$, and $N \in \mathcal{S}$ such that

$$f_n(x) + \varepsilon > \sup_{y \in V} f_n(y)$$

for all $n \in N$. This implies

$$\text{st-}\liminf_n f_n(x) + \varepsilon \geq \inf_{V \in \mathcal{N}(x)} \text{st-}\liminf_n \sup_{y \in V} f_n(y) = h_{st}\text{-}\liminf_n f_n(x)$$

for every $\varepsilon > 0$. Combining with Proposition 2.7 we get

$$\text{st-}\liminf_n f_n(x) = \inf_{V \in \mathcal{N}(x)} \text{st-}\liminf_n \sup_{y \in V} f_n(y)$$

which means,

$$\text{st-}\liminf_n f_n(x) = h_{st}\text{-}\liminf_n f_n(x).$$

In similar way, we get st- lim $\sup_n f_n(x) = h_{st}$- lim $\sup_n f_n(x)$ and finally we reach the desired equality as follows

$$\text{st-}\lim_n f_n(x) = h_{st}\text{-}\lim_n f_n(x).$$

3. Conclusion

The results in this paper which we derived from statistical Kuratowski limits will be used for our further studies related with maximizers of statistical lower and upper hypo-limit functions. Since, statistical hypo-convergence of sequence of functions is important for maximization of stochastic optimization and variational problems.

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