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Hamiltonian Equations On Kähler-Einstein Fano-Weyl Manifolds

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Abstract

The paper aims to introduce Hamiltonian equations for mechanical systems using Kähler angles on Kähler-Einstein Fano-Weyl manifold which represent an interesting multidisciplinary field of research. Also, solutions of these equations will be made using the computer program with Maple and the geometrical-physical results related to on Kähler-Einstein Fano-Weyl mechanical systems are also to be issued.

Keywords

Fano, Kähler-Einstein, Hamiltonian, Mechanical System.

1. Introduction

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor is proportional to the metric for differential geometry and mathematical physics. They are named after Einstein because this condition is equivalent to saying that the metric is a solution of the vacuum Einstein field equations, although the dimension, as well as the signature, of the metric can be arbitrary, unlike the four-dimensional Lorentzian manifolds usually studied in general relativity. A Kähler-Einstein metric on a complex manifold is a Riemannian metric that is both a Kähler metric and an Einstein metric. A manifold is said to be Kähler-Einstein if it admits a Kähler-Einstein metric. The most important problem for this area is the existence of Kähler-Einstein metrics for compact Kähler manifolds. In the case in which there is a Kähler metric, the Ricci curvature is proportional to the Kähler metric. Therefore, the first Chern class is either negative, or zero, or positive. A Fano manifold is a compact Kähler manifold with positive first Chern class. Tian introduced that Kähler-Einstein metrics on K-stable Fano manifolds and Kähler metrics of positive scalar curvature (Tian, 2015). Tian discussed some of my works on the existence of Kähler-Einstein metrics on Fano manifolds and some related topics (Tian, 2012). Arezzoa and Nave shown that the existence of stable symplectic non-holomorphic two-spheres in Kähler manifolds of positive constant scalar curvature of real dimension four and in Kähler-Einstein Fano manifolds of real dimension six (Arezzoa and Naveb, 2005). Chen et al announced final paper in a series of Kähler-Einstein metrics and stability (Chen et al, 2013). Li et al. investigated the geometry of the orbit space of the closure of the subschema parametrizing smooth Fano Kähler-Einstein manifolds inside an appropriate Hilbert scheme (Li et al, 2014). Suss applied this to certain Fano varieties and use Tian's criterion to prove the existence of Kähler-Einstein metrics on them (Suss, 2012).

The Weyl curvature tensor is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. Folland presented the general properties of Weyl manifolds (Folland, 1970). Hall studied of Weyl connections and their associated holonomy groups (Hall, 1992). Kadosh made a PhD thesis study on Weyl geometry an he also offered a comparison between Weyl and Riemann geometry (Kadosh, 1996). Kasap and Tekkoyun introduced Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler-Weyl manifolds (Kasap, 2013). Han et al. gave a classification of conformal-Weyl manifolds based on the perspective of semi-symmetric non-metric connections (Han et al, 2016).

2. Preliminaries

Definition 1. Let M be a differentiable manifold of dimension 2n, and suppose J is a differentiable vector bundle isomorphism $J:TM\to TM$ such that $J_p:T_pM\to T_pM$ is a complex structure for T_pM, i.e. $J^2=-I$ ($J^2=I$ or $J^2=0$), where $J^2=J\circ J$ and I is the identity vector bundle isomorphism. Then J is called an almost-complex (paracomplex or tangent) structure for the differentiable manifold M. A manifold with a fixed almost-complex structure is called an almost-complex (paracomplex or tangent) manifold.
Definition 2. If M is a smooth manifold of real dimension 2n, then a smooth field $J=(J_x)$ of complex structures on TM is called an almost complex structure of M. An almost complex structure $J=J_x$ is called a complex structure if it comes from a complex structure on M as in $J_xX_j(x)=Y_j(x)$, $J_xY_j(x)=-X_j(x)$. Any almost complex structure on a surface is a complex structure.
Definition 3. Let M be a smooth manifold of dimension $n\geq 3$. Let ∇ be its Levi-Civita connection, a torsion free connection on the tangent bundle TM of M and let g=<.,.> be a pseudo-Riemann metric on M of signature (p,q). (M,g) be called the pseudo-Riemannian manifold (Gilkey, 2010). The Ricci (curvature) tensor r of a pseudo-Riemannian manifold (M,g) is the 2-tensor $r(X,Y)=tr(Z)$,where tr donates the trace of the linear map $Z\to R(X,Z)Y$. Note that the Ricci tensor is symmetric.
Remark 1. Let $z_1,...,z_n$ be coordinates on $\mathbb{C}^n$. Write $z_j=x_j+\mathbf{i}y_j$. The $x_1,...,x_n$ and $y_1,...,y_n$ are real coordinates on $\mathbb{C}^n$. For $p\in\mathbb{C}^n$, the tangent space $T_p\mathbb{C}^n$ has a basis $\partial/\partial x_1,...,\partial/\partial x_n,\partial/\partial y_1,...,\partial/\partial y_n$ and the cotangent space $T_p{*}\mathbb{C}^n$ has a basis $dx_1,...,dx_n,dy_1,...,dy_n$.
Remark 2. Let TM be an almost complex manifold with fixed almost complex structure J and TM is called complex manifold. $J:T_p\mathbb{C}^n\to T_p\mathbb{C}^n$ by

$$J(\partial/(\partial x_j))=\partial/(\partial y_j),$$
$$J(\partial/(\partial y_j))=-\partial/(\partial x_j). \tag{1}$$

Then $J^T(dx_j)=-dy_j$ and $J^T(dy_j)=dx_j$. A basis for the $+\mathbf{i}$ eigenspace for J^T is $dz_1,...,dz_n$ where $dz_j=dx_j+\mathbf{i}dy_j$, $d\bar{z}_j=dx_j-\mathbf{i}dy_j$. So

$$J^T(z_j)=J^T(dx_j+\mathbf{i}dy_j)=-dy_j+\mathbf{i}dx_j=\mathbf{i}(dx_j+\mathbf{i}dy_j)=\mathbf{i}z_j. \tag{2}$$

A basis for the $-\mathbf{i}$ eigenspace for J^T is $d\bar{z}_j=dx_j-\mathbf{i}dy_j$. The dual basis to $dz_1,...,dz_n,dz_1,...,dz_n$ of $T_p{*}\mathbb{C}^n\otimes\mathbb{C}$ is $\partial/\partial z_1,...,\partial/\partial z_n,\partial/\partial z_1,...,\partial/\partial z_n$, where

$$\partial/(\partial z_j)=(1/2)(\partial/(\partial x_j)-\mathbf{i}\partial/(\partial y_j)),$$
$$\partial/(\partial\bar{z}_j)=(1/2)(\partial/(\partial x_j)+\mathbf{i}\partial/(\partial y_j)). \tag{3}$$

Then the endomorphism J of the tangent space $T_p(TM)$ at any point p of manifold TM satisfies $J^2=-I$ and is defined by

$$J(\partial/(\partial z_j))=\mathbf{i}(\partial/(\partial z_j)),$$
$$J(\partial/(\partial\bar{z}_j))=-\mathbf{i}(\partial/(\partial\bar{z}_j)). \tag{4}$$

The dual endomorphism J of the cotangent space $T_p{*}(TM)$ at any point p of manifold TM satisfies $J{*}^2=-I$ and is defined by

$$J(dz_j)=\mathbf{i}dz_j,$$
$$J(d\bar{z}_j)=-\mathbf{i}d\bar{z}_j. \tag{5}$$

A Riemannian manifold (M,g) consists of the following data: a compact C^∞ manifold M. A metric tensor field g which is a positive definite bilinear symmetric differential form on M. In other words, we associate with every point p of M a Euclidean structure g_p on the tangent space T_pM of M at p and require the association $p\to g_p$ to be C^∞. We say that g is a Riemannian metric on M. A pseudo-Riemannian manifold (also called a semi-Riemannian manifold) (M,g) is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g. Also, it is generalization of a Riemannian manifold in which the metric tensor need not be positive-definite. A pseudo-Riemannian manifold (M,g) is Einstein manifold if there exists a real constant λ such that

$$r(X,Y)=\lambda_g(X,Y), \tag{6}$$

for $\forall p\in M$, $\forall X,Y\in T_pM$.

Theorem 1. Assume $n\geq 3$. Then an n-dimensional pseudo-Riemannian manifold is Einstein if and only if, for each p in M, there exists a constant λ_p such that

$$r_p=\lambda_p g_p. \tag{7}$$

Proof: The "only if" part is trivial. In the other direction, applying the divergence δ to both sides of (7), we get $\delta r=-(1/2)ds=-d\lambda$. So $\lambda-(1/2)s$ is a constant. Taking the trace of (7) with respect to g, we get $n\lambda=s$. So finally λ (and s) are constant (Besse, 1987).

A pseudo-holomorphic curve (J-holomorphic curve) is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchy-Riemann equation (McDu and Salamon, 1995). A closed two-form ω on a complex manifold M which is also the negative imaginary part of a Hermitian metric $h=g-\mathbf{i}\omega$ is called a Kähler form. In this case, M is called a Kähler manifold and g, the real part of the Hermitian metric, is called a Kähler metric. The Kähler form combines the metric and the complex structure, indeed

$$g(X,Y)=\omega(X,JY), \tag{8}$$

where J is the almost complex structure. Since the Kähler form comes from a Hermitian metric, it is preserved by J, i.e., since $h(X,Y)=h(JX,JY)$. The equation $d\omega=0$ implies that the metric and the complex structure are related. It gives M a Kähler structure and has many implications. A Kähler metric g on a complex manifold M is Einstein if and only if there exists $\lambda\in\mathbb{R}$ such that

$$\rho=\lambda\omega, \tag{9}$$

where ω is the fundamental form associated to g and

$$\rho(X,Y)=Ric(X,JY), \tag{10}$$

for $X,Y\in\chi(M)$. The pair (M,g), where M is a complex manifold and g a Kähler-Einstein metric is said a Kähler-Einstein manifold (Zedda, 2009). A Kähler-Einstein metric on a complex manifold is a Riemannian metric that is both a Kähler metric and an Einstein metric. A manifold is said to be Kähler-Einstein if it admits a Kähler-Einstein metric. A Kähler metric on a complex manifold whose Ricci tensor $Ric(\omega)$ is proportional to the metric tensor:

$$Ric(\omega)=\lambda\omega. \tag{11}$$

Let M be a complex manifold with complex structure J and compatible Riemannian metric $g=\langle .,. \rangle$ as in $\langle JX,JY\rangle=\langle X,Y\rangle$, where X and Y any two vector fields. The alternating 2-form

$$\omega(X,Y):=g(JX,Y), \tag{12}$$

is called the associated Kähler form. We can retrieve g from ω,

$$g(X,Y)=\omega(X,JY). \tag{13}$$

We say that g is a Kähler metric and that M is a Kähler manifold if ω is closed and (M,g) is displayed in the form. Let M be a complex manifold. A Riemannian metric on M is called Hermitian if it is compatible with the complex structure J of M,

$$\langle JX,JY\rangle=\langle X,Y\rangle. \tag{14}$$

Then the associated differential two-form ω defined by

$$\omega(X,Y)=\langle JX,Y\rangle, \tag{15}$$

is called the Kähler form. It turns out that ω is closed if and only if J is parallel. Then M is called a Kähler manifold and the metric on M a Kähler metric. Kähler manifolds are modelled on complex Euclidean space.

Definition 4. Let M be a compact connected complex manifold and $c_1(M)_{\mathbb{R}}$ its first Chern class; if $c_1(M)_{\mathbb{R}}>0$, M is Fano manifold, then M carries a unique (Ricci-positive) Kähler-Einstein metric ω such that for $\lambda=1$,

$$Ric(\omega)=\omega. \tag{16}$$

In algebraic geometry, a Kähler manifold M with $c_1(M)_{\mathbb{R}}>0$ is called a Fano manifold (Wang, 2004).

3. The Kähler Angle

The principal or canonical angles (and the related principal vectors) between two subspaces provide the best available characterization of the relative subspace positions. In any (finite-dimensional) real (Euclidean) vector space $V_{\mathbb{R}}(\simeq\mathbb{R}_m, m\in\mathbb{N}, m\geq 2)$ equipped with the scalar product $<X,Y>_{\mathbb{R}}=\sum_{k=1}^{m} X_k Y_k$ for any pair of vectors $X,Y\in V_{\mathbb{R}}$ one can define an (real) angle θ, $0\leq\theta\leq\pi$, between these two vectors by means of the standard formula

$$\cos\theta=(<X,Y>_{\mathbb{R}})/(\|X\|\|Y\|). \tag{17}$$

The Kähler Angle: In order to proceed further let us introduce the almost complex structure J, $J^2=-I$, which acts as an operator in the real vector space $V_{\mathbb{R}}$ isometric to $V_{\mathbb{C}}$. In our coordinates the almost complex structure J performs the following transformations: $X_{2k-1}\rightarrow X_{2k}$, X_{2k}-X_{2k-1}, k=1,...,n. This is equivalent to the transformation $x\rightarrow \mathbf{i}x$ in $V_{\mathbb{C}}$. A subspace P of $V_{\mathbb{R}}$ is called holomorphic, if it holds P=JP. It is called antiholomorphic (totally real, with a real Hermitian product), if it holds $P\perp JP$. Following the convention applied in a large fraction of the literature we introduce the notation $\widetilde{X}=JX$, $X\in V_{\mathbb{R}}$. By writing

$$\cos\theta_K\sin\theta=(<X,Y>_{\mathbb{R}})/(\|X\|.\|Y\|), \tag{18}$$

one can now introduce a further angle $0\leq\theta_K\leq\pi$, which is called the Kähler Angle between the vectors $x,y\in V_{\mathbb{C}}$ or the vectors $X,Y\in V_{\mathbb{R}}$, respectively (Scharnhorst, 2001).

Definition 5. Let N be a Kähler manifold with the complex structure J and the standard Kähler metric <.,.>, let M be a Riemann surface; and let $\Psi:M\rightarrow N$ be an isometric minimal immersion of M into N. Then the Kähler angle θ of Ψ which is an invariant of the immersion Ψ related to J, is defined by

$$\cos\theta=<Je_1,e_2>. \tag{19}$$

where $\{e_1,e_2\}$ is an orthonormal basis of M (Mo, 1994).

4. Conformal and Weyl Structure

Definition 6. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics g′ and g are equivalent if and only if

$$g'=\Psi^2 g \tag{20}$$

where Ψ>0 is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class. Two Riemann metrics g_1 and g_2 on M are said to be conformally equivalent if there exists a smooth function $f:M\rightarrow\mathbb{R}$ with

$$e^f g_1=g_2. \tag{21}$$

In this case, $g_1\sim g_2$.

Definition 7. Let M an n-dimensional smooth manifold. A pair (M,C), a conformal structure on M is an equivalence class C of Riemann metrics on M, is called a conformal structure.

Theorem 2. Let ∇ be a connection on M and g∈C a fixed metric. ∇ is compatible with (M,C)⇔ there exists a 1-form ω with $\nabla_X g+\omega(X)g=0$ (proof see Kadosh, 1996).

Definition 8. A compatible torsion-free connection is called a Weyl connection. The triple (M, C, ∇) is a Weyl structure.

Theorem 3. To each metric g∈C and 1-form ω, there corresponds a unique Weyl connection ∇ satisfying $\nabla_X g+\omega(X)g=0$ (proof see Kadosh, 1996).

5. Weyl Geometry

Definition 9. Let M be a smooth manifold of dimension n≥3. Let ∇ be a torsion free connection on the tangent bundle TM of M and let g be a pseudo-Riemann metric on M of signature (p, q). The triple (M, g, ∇) is said to be a Weyl manifold if there exists a smooth 1-form $\phi_{\nabla,g}\in C^\infty(T^*M)$ so that $\nabla g=-2\phi_{\nabla,g}\otimes g$ (Gilkey and Nikcevic, 2010).

Definition 10. Let (M, g, ∇, $J_\pm$) be an almost pseudo (J_-)/para(J_+)-Hermitian Weyl manifold. If $\nabla(J_\pm)=0$, then one says that this is a (para)-Kähler Weyl manifold.

Theorem 4. Let m≥6. If (M, g, ∇, $J_\pm$) is a Kähler-Weyl structure, then the associated Weyl structure is trivial, i.e. there is a conformally equivalent metric,

$$\tilde{g}=e^{2f}g \tag{22}$$

so that (M, g, $J_{\pm}$) is Kähler and so that $\nabla=\nabla^{\tilde{g}}$ (proof see , Gilkey and Nikcevic, 2010).

Definition 11. The Weyl transformation is a local rescaling of the metric tensor:

$$g_{ab}\rightarrow e^{-2\omega(x)}g_{ab}, \tag{23}$$

that produces another metric in the same conformal class. A theory or an expression invariant under this transformation is called conformally invariant, or is said to possess Weyl symmetry (Weyl, 1921).

6. Properties of F*ω

Let (N, J, g) be a Kähler manifold of complex dimension 2n and g is a Kähler metric. Also, F:M→N an immersed submanifold of real dimension 2n and minimal submanifold M. We denote by ω the Kähler form and $x,y\in\chi(M)$:

$$\omega(x,y)=g(Jx,y). \tag{24}$$

We take the induced metric on M

$$g_{\mu}=F^{*}g. \tag{25}$$

N is Kähler-Einstein manifold if its Ricci tensor is a multiple of the metric, $Ricci^{N}=Rg$. At each point p∈M, we identify $F^{*}\omega$ with a skew-symmetric operator of T_pM by using the musical isomorphism with respect to g_{μ} namely

$$g_{\mu}(F^{*}\omega(x),y)=F^{*}\omega(x,y). \tag{26}$$

We take its polar decomposition

$$F^{*}\omega=gJ_{\omega}, \tag{27}$$

where $J_{\omega}:T_pM\rightarrow T_pM$ is a partial isometry with the same kernel κ_{ω} as of $F^{*}\omega$, and where g is the positive semi-definite operator

$$g=|F^{*}\omega|=\sqrt{-F^{*2}\omega}. \tag{28}$$

Let's take a Kähler-Einstein metric g. If X and Y are orthonormal basis on M then cosθ=<JX,Y> according to (17) and (19). Also, ω(X,Y)=g(JX,Y)=<JX,Y> at (15) and (12). ρ=Ric(ω)=ω for first Chern class (λ=1) (9):

$$\rho=Ric(\omega)=\lambda\omega(X,Y)=g(JX,Y)=<JX,Y>=\cos\theta. \tag{29}$$

We take equation (29) into consideration (27) then $F^{*}\omega$ is as follows:

$$F^{*}\omega=\cos\theta J_{\omega} \tag{30}$$

Let $\{x_{\alpha},y_{\alpha}\}_{\{1\leq\alpha\leq n\}}$ be a g_{μ}-orthonormal basis of T_pM, that diagonalizes $F^{*}\omega$ at p, that is

$$F^{*}\omega\begin{bmatrix}y_{\alpha}\\x_{\alpha}\end{bmatrix}=\bigoplus_{0\leq\alpha\leq n}\begin{bmatrix}0 & -\cos\theta_{\alpha}\\ \cos\theta_{\alpha} & 0\end{bmatrix}\begin{bmatrix}x_{\alpha}\\y_{\alpha}\end{bmatrix} \tag{31}$$

where $\cos\theta_1\geq\cos\theta_2\geq\cdots\geq\cos\theta_n\geq0$. The angles $\{\theta_{\alpha}\}_{1\leq\alpha\leq n}$ are the Kähler angles of F at p. Thus, using (30) for ∀α,

$$F^{*}\omega(x_{\alpha})=\cos\theta_{\alpha}y_{\alpha},$$
$$F^{*}\omega(y_{\alpha})=-\cos\theta_{\alpha}x_{\alpha}, \tag{32}$$

and if k≥1, where 2k is the rank of $F^{*}\omega$ at p, $J\omega(x_{\alpha})=y_{\alpha}$,∀α≤k. M is a complex submanifold iff $\cos\theta_{\alpha}=1$, ∀α, and is a Lagrangian submanifold iff $\cos\theta_{\alpha}=0$, ∀α. We say that F has equal Kähler angles if $\theta_{\alpha}=\theta$, ∀α. Complex and Lagrangian submanifolds are examples of such case. If F is a complex submanifold, then J is the complex structure induced by J of N. The Kähler angles are some functions that at each point p of M measure the deviation of the tangent plane T_pM of M from a complex or a Lagrangian subspace of $T_{F(p)}M$. This concept was introduced by Chern and Wolfson for oriented surfaces, namely $F_{*}\omega=\cos\theta Vol_M$ (Salavessa and Valli, 2002).

Theorem 5. If M is a real compact surface and N is a complex Kähler-Einstein surface with R<0, and if F is minimal with no complex points, then F is Lagrangian (proof see Wolfson, 1989).

Let we denote by $\nabla_x dF(y)=\nabla dF(x,y)$ the second fundamental form of F. If F is an immersion with no complex directions at p and $\{x_{\alpha},y_{\alpha}\}$ diagonalizes $F^{*}\omega$ at p, then $\{dF(z_{\alpha}),dF(\bar{z}_j),(JdF(z_{\alpha}))^{\perp},(JdF(\bar{z}_j))^{\perp},($ constitutes a complex basis of $T^{\mathbb{C}}_{F(p)}N$, where **i** ^{2}=-1,

$$z_\alpha=(1/2)(x_\alpha-\mathbf{i}y_\alpha),$$
$$\bar{z}_\alpha=(1/2)(x_\alpha+\mathbf{i}y_\alpha), \quad (33)$$

are complex vectors of the complexfield tangent space of M at p. If F has equal Kähler angles, then

$$F^*\omega=\cos\theta J_\omega,$$
$$\tilde{g}=\sin^2\theta g_\mu. \quad (34)$$

If we parallel transport a diagonalizing orthonormal basis $\{x_\alpha,y_\alpha\}$ of $F^*\omega$ at p_0 along geodesics, on a neighborhood of p_0. Similarly we that , $\tilde{g}$ is parallel. If we extend $F^*\omega$ to the complexified tangent space $T^{\mathbb{C}}_{p_0}M$ then the holomorphic base structures, considering (32), (33) and (34), are as follows (Salavessa and Valli, 2002):

$$1.\ F^*\omega(z_\alpha)=\mathbf{i}\cos\theta_\alpha\bar{z}_\alpha,$$
$$2.\ F^*\omega(\bar{z}_\alpha)=-\mathbf{i}\cos\theta_\alpha z_\alpha. \quad (35)$$

M is a paracomplex submanifold iff $\theta_\alpha=2k\pi$, $k\in\mathbb{Z}$. These structures (35) can be edited using the properties of Weyl geometry (22) and (23).

Theorem 6. Suppose that $\{z_\alpha,\bar{z}_\alpha\}$, be a complex coordinate system on paracomplex M manifold. Then the bases cotangent space for $dz_\alpha\rightarrow z_\alpha$ and $d\bar{z}_\alpha\rightarrow\bar{z}_\alpha$,

$$1.\ F^*\omega(z_\alpha)=\mathbf{i}\cos\theta_\alpha e^{2f}\bar{z}_\alpha,$$
$$2.\ F^*\omega(\bar{z}_\alpha)=-\mathbf{i}\cos\theta\alpha e^{-2f}z_\alpha. \quad (36)$$

If $F^*\omega$ is defined as a paracomplex on Kähler-Einstein Fano-Weyl manifolds then $F^{*2}\omega=F^*\omega\circ F^*\omega=I$, $\theta_\alpha=2k\pi$, $k\in\mathbb{Z}$.

Proof: Let's find the structure property using Definition 1.

$$\begin{aligned}1.\quad F^{*2}\omega(z_\alpha)&=F^*\omega\circ F^*\omega(\mathbf{i}\cos\theta_\alpha e^{2f}\bar{z}_\alpha),\\&=\mathbf{i}\cos\theta_\alpha e^{2f}F^*\omega(\bar{z}_\alpha),\\&=-\mathbf{i}^2\cos^2\theta_\alpha z_\alpha,\\2.\quad F^{*2}\omega(\bar{z}_\alpha)&=F^*\omega\circ F^*\omega(-\mathbf{i}\cos\theta_\alpha e^{-2f}z_\alpha),\\&=-\mathbf{i}\cos\theta_\alpha e^{-2f}F^*\omega(z_\alpha),\\&=-\mathbf{i}^2\cos^2\theta_\alpha\bar{z}_\alpha.\end{aligned} \quad (37)$$

As we have seen above, these structures (36) have the ability to paracomplex for $\theta_\alpha=2k\pi$, $k\in\mathbb{Z}$. Because, $F^2\omega(z_\alpha)=z_\alpha$, $F^2\omega(\bar{z}_\alpha)=\bar{z}_\alpha$ so $F^2\omega=I$.

7. Hamiltonian Dynamics Equations

Let M is the configuration manifold and its cotangent manifold T^*M. By a symplectic form we mean a 2-form Φ on T^*M such that (i) Φ is closed that is, $d\Phi=0$, (ii) $\forall z\in T^*M$, $\Phi:T^*M\times T^*M\rightarrow\mathbb{R}$. Let (T^*M,Φ) be a symplectic manifold. A vector field $Z_H:T^*M\rightarrow T^*M$ is called Hamiltonian if there is a C^1function $H:T^*M\rightarrow\mathbb{R}$ such that dynamical equation is determined by

$$i_{Z_H}\Phi=dH. \quad (38)$$

Z_H is locally Hamiltonian vector field if $i_{Z_H}\Phi$ is closed and where Φ shows the canonical symplectic form so that $\Phi=-d\Omega$, $\Omega=J^*(\omega)$, J^* a dual of J, ω a 1-form on T^*M. The trio (T^*M,Φ,Z_H) is named Hamiltonian system which it is defined on the cotangent bundle T^*M and $Z_H(\alpha(t)=\alpha(t)$ (Klein, 1962).

Hamilton's equations are an alternative to the Euler-Lagrange equations to find the equations of motion of a system. They are applied to the Hamiltonian function of the system. Hamilton's equations are

$$\dot{q}_i=(\partial H)/(\partial p_i),$$
$$\dot{p}_i=-(\partial H)/(\partial q_i),\ (\partial L)/(\partial t)=-(\partial H)/(\partial t). \quad (39)$$

Where H is the Hamiltonian, q_i are the generalized coordinates, p_i are the generalized momenta, and a dot represents the total time derivative. The Hamiltonian can be found by performing a Legendre transformation on the Lagrangian of the system: $H=\dot{q}_ip_i-L(q,\dot{q},t)$, where the Einstein summation notation is used, a dot represents the total time derivative, q_i are the generalized coordinates, p_i are the generalized momenta. These momenta are found by differentiating the Lagrangian with respect to the generalized velocities $\dot{q}_i$. Mathematically $p_i=(\partial L)/(\partial\dot{q})$. H=T+V is the total energy of the system. There are two conditions for this. First, the equations defining the generalized coordinates q don't depend on time explicitly. Second, the forces involved in the system are derivable from a scalar potential. The forces must be conservative (such as gravity) (Goldstein et al, 2002).

8. Hamiltonian Equations

We, using (38), will present Hamilton equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on Kähler-Einstein Fano-Weyl manifolds.

Proposition: Let $(M,g,F^*\omega)$ be on Kähler-Einstein Fano-Weyl manifolds. Suppose that the complex structures, a Liouville form and a 1-form on Kähler-Einstein Fano-Weyl manifolds with equal Kähler angles are shown by $F^*\omega$, Ω and ξ, respectively. Consider a 1-form ξ on (36) such that

$$\xi=(1/2)[z_\alpha dz_\alpha+\bar{z}_\alpha d\bar{z}_\alpha]. \quad (40)$$

The Hamiltonian partial differential equations of the system are as follows:

$$1.\ (\partial z_\alpha)/(\partial t)=((2\mathbf{i})/(\cos\theta_\alpha[-2((\partial f)/(\partial z_\alpha))e^{-2f}z_\alpha-e^{-2f}]))((\partial H)/(\partial\bar{z}_\alpha)),$$

$$2.\ (\partial\bar{z}_\alpha)/(\partial t)=((2\mathbf{i})/(\cos\theta_\alpha[2((\partial f)/(\partial\bar{z}_\alpha))e^{2f}\bar{z}_\alpha+e^{2f}]))((\partial H)/(\partial z_\alpha)). \quad (41)$$

Proof: We obtain the Liouville form as follows:

$$\begin{aligned}\Omega&=F^*\omega(\xi)\\&=1/2[F^*\omega(z_\alpha)dz_\alpha+F^*\omega(\bar{z}_\alpha)d\bar{z}_\alpha],\\&=(1/2)[\mathbf{i}\cos\theta_\alpha e^{2f}\bar{z}_\alpha dz_\alpha-\mathbf{i}\cos\theta_\alpha e\text{-}2fz_\alpha d\bar{z}_\alpha],\\&=((\mathbf{i}\cos\theta_\alpha)/2)[e^{2f}\bar{z}_\alpha dz_\alpha-e^{-2f}z_\alpha d\bar{z}_\alpha].\end{aligned} \quad (42)$$

It is well known that if Φ is a closed on Kähler-Einstein Fano-Weyl manifolds with equal Kähler angles $(M,g,F^*\omega)$, then Φ is also a symplectic structure on $(M,g,F^*\omega)$. Therefore the 2-form $\Phi=-d\Omega$ indicates the canonical symplectic form and derived from the 1-form Ω to find to mechanical equations. Then the 2-form Φ is calculated as below:

$$\begin{aligned}\Phi=-d\Omega=-((\mathbf{i}\cos\theta_\alpha)/2)[&\ [2((\partial f)/(z_\alpha))e^{2f}\bar{z}_\alpha]dz_\alpha\wedge dz_\alpha\\&-[-2((\partial f)/(z_\alpha))e^{-2f}z_\alpha-e^{-2f}]dz_\alpha\wedge d\bar{z}_\alpha\\&+[2((\partial f)/(\bar{z}_\alpha))e^{2f}\bar{z}_\alpha+e^{2f}]d\bar{z}_\alpha\wedge dz_\alpha\\&-[-2((\partial f)/(\bar{z}_\alpha))e^{-2f}z\alpha]d\bar{z}_\alpha\wedge d\bar{z}_\alpha].\end{aligned} \quad (43)$$

Take a vector field Z_H so that called to be Hamiltonian vector field associated with Hamiltonian energy H and determined by

$$Z_H=Z^\alpha(\partial/(z_\alpha))+\bar{Z}^\alpha(\partial/(\bar{z}_\alpha)). \quad (44)$$

So, we have

$$\begin{aligned}i_{Z_H}\Phi&=\Phi(Z_H)\\&=-((\mathbf{i}\cos\theta_\alpha)/2)[Z^\alpha[2((\partial f)/(z_\alpha))e^{2f}\bar{z}_\alpha](dz_\alpha((\partial/(z_\alpha)))dz_\alpha-dz_\alpha((\partial/(z_\alpha)))dz_\alpha)\\&\quad-Z^\alpha[-2((\partial f)/(z_\alpha))e^{-2f}z_\alpha-e^{-2f}](dz_\alpha((\partial/(z_\alpha)))d\bar{z}_\alpha-d\bar{z}_\alpha((\partial/(z_\alpha)))dz_\alpha)\\&\quad+\bar{Z}^\alpha[2((\partial f)/(\bar{z}_\alpha))e^{2f}\bar{z}_\alpha+e^{2f}](d\bar{z}_\alpha((\partial/(z_\alpha)))dz_\alpha-dz_\alpha((\partial/(z_\alpha)))d\bar{z}_\alpha)\\&\quad-\bar{Z}^\alpha[-2((\partial f)/(\bar{z}_\alpha))e^{-2f}z_\alpha](d\bar{z}_\alpha((\partial/(z_\alpha)))d\bar{z}_\alpha-d\bar{z}_\alpha((\partial/(z_\alpha)))d\bar{z}_\alpha)]\end{aligned} \quad (45)$$

Furthermore, the differential of Hamiltonian energy H is obtained by

$$dH=((\partial H)/(z_\alpha))dz_\alpha+((\partial H)/(\bar{z}_\alpha))d\bar{z}_\alpha. \quad (46)$$

From $i_{Z_H}\Phi=\Phi(Z_H)$, the Hamiltonian vector field is found as follows:

$$\bar{Z}^\alpha=((-2)/(\mathbf{i}\cos\theta_\alpha[2((\partial f)/(\bar{z}_\alpha))e^{2f}\bar{z}_\alpha+e^{2f}]))((\partial H)/(z_\alpha)),$$

$$Z^\alpha=((-2)/(\mathbf{i}\cos\theta_\alpha[-2((\partial f)/(z_\alpha))e^{-2f}z_\alpha-e^{-2f}]))((\partial H)/(\bar{z}_\alpha)), \quad (47)$$

and then

$$\begin{aligned}Z_H=&((-2\mathbf{i})/((\cos\theta_\alpha[-2((\partial f)/(z_\alpha))e^{-2f}z_\alpha-e^{-2f}]))((\partial H)/(\bar{z}_\alpha))(\partial/(z_\alpha))\\&+((2\mathbf{i})/(\cos\theta_\alpha[2((\partial f)/(z_\alpha))e^{2f}\bar{z}_\alpha+e^{2f}]))((\partial H)/(z_\alpha))(\partial/(\bar{z}_\alpha)).\end{aligned} \quad (48)$$

Consider the curve and its velocity vector

$$\alpha: I\subset\mathbb{R}\rightarrow M,\ \alpha(t)=(z_\alpha,\bar{z}_\alpha),$$
$$\dot{\alpha}(t)=(\partial\alpha)/(\partial t)=((dz_\alpha)/(dt),(d\bar{z}_\alpha)/(dt)), \quad (49)$$

such that an integral curve of the Hamiltonian vector field Z_H, i.e.,

$$\dot{\alpha}(t)=((d\alpha)/(dt))=((dz_\alpha)/(dt))(\partial/(z_\alpha))+((d\bar{z}_\alpha)/(dt))(\partial/(\bar{z}_\alpha)),\ t\in I. \quad (50)$$

Then, $Z_H(\alpha(t))=\dot{\alpha}(t)$,

$$((2\mathbf{i})/(\cos\theta_\alpha[-2((\partial f)/(z_\alpha))e^{-2f}z_\alpha-e^{-2f}]))((\partial H)/(\bar{z}_\alpha))(\partial/(z_\alpha))$$
$$+((2\mathbf{i})/(\cos\theta_\alpha[2((\partial f)/(z_\alpha))e^{2f}\bar{z}_\alpha+e^{2f}]))((\partial H)/(z_\alpha))(\partial/(\bar{z}_\alpha))$$
$$=((z_\alpha)/(\partial t))(\partial/(z_\alpha))+(\bar{z}_\alpha)/(\partial t))(\partial/(\bar{z}_\alpha)). \quad (51)$$

We find the following equations;

$$1.\ (\partial z_\alpha)/(\partial t)=((2\mathbf{i})/(\cos\theta_\alpha[-2((\partial f)/(\partial z_\alpha))e^{-2f}z_\alpha-e^{-2f}]))((\partial H)/(\partial\bar{z}_\alpha)),$$
$$2.\ (\partial\bar{z}_\alpha)/(\partial t)=((2\mathbf{i})/(\cos\theta_\alpha[2((\partial f)/(\partial\bar{z}_\alpha))e^{2f}\bar{z}_\alpha+e^{2f}]))((\partial H)/(\partial z_\alpha)). \quad (52)$$

Hence, the equations introduced in (55) are named Hamiltonian equations with Kähler angles on Kähler-Einstein Fano-Weyl manifolds and then the triple (M,Φ,Z_H) is said to be a Hamiltonian Mechanical System on Kähler-Einstein Fano-Weyl manifolds.

9. Solution of Hamilton Equations

Those found (41) are partial differential equation. The equations system (41) have be solved by using the symbolic Maple Algebra computer program.

$$H(z_\alpha,\bar{z}_\alpha,t)=(-2*(\cos(t)-\mathbf{i}*\sin(t))*\cos(2*t)+2*\mathbf{i}*(\cos(t)-\mathbf{i}*\sin(t))*\sin(2*t)$$
$$-2*\exp^4(t)*(\cos(t)+\sin(t)*\mathbf{i}-t))/(\sin(2*t)*\mathbf{i}-\cos(2*t))/\exp(t)/(\cos(t)+\sin(t)*\mathbf{i}). \quad (53)$$

First, implicit solutions of partial differential equation with the help of Maple computer program at (53) will be selected as a special for $z_\alpha=\cos(t)-\mathbf{i}\sin(t)$, $\theta_\alpha=0$, $f=t$. After, the graph of the equation (53) has been drawn for the route of the movement of objects in the electromagnetic field as follows;

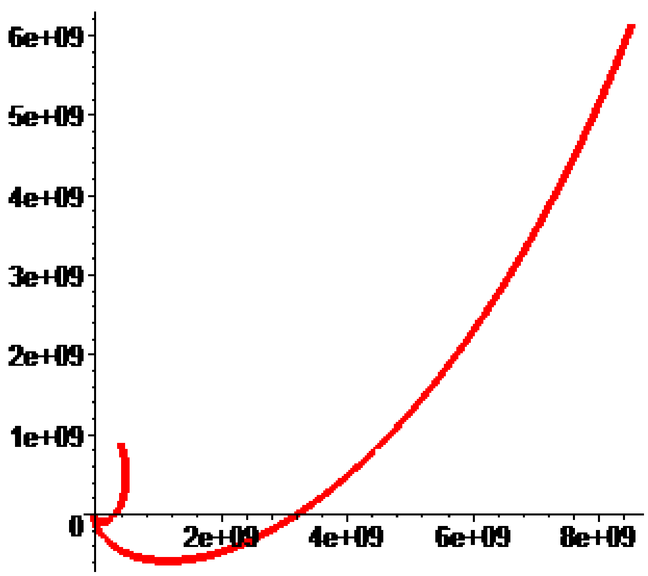

Figure 1

10. Discussion

A classical field theory explains the study of how one or more physical fields interact with matter which is used quantum and classical mechanics of physics branches (Deleon and Rodrigues, 1989). It is well-known that an electromagnetic field is a physical field produced by electrically charged objects. How the movement of objects in electrical, magnetically and gravitational fields force is very important. For instance, on a weather map, the surface wind velocity is defined by assigning a vector to each point on a map. So, each vector represents the speed and direction of the movement of air at that point. In this study, the Hamiltonian equations derived (41) with Kähler angles on Kähler-Einstein Fano-Weyl manifolds for mechanical systems and a special solution of the equation system was found (53). Also, they may be suggested to deal with problems in electrical, magnetically and gravitational fields force for the path of movement (Figure 1) of defined space moving objects (Thide, 2004).

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