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Necessary and Sufficient Conditions for Oscillation of Nonlinear Second-Order Delay Differential Equations

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Abstract

In this work, necessary and sufficient conditions are obtained by using Lebesque's Dominated Convergence theorem for oscillation of solutions of second-order delay differential equations of the form:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[a(t)\frac{\mathrm{d}}{\mathrm{dt}}x(t)\right]+q(t)H\big(x(t-\sigma)\big)=0, \qquad \mathrm{t}>\mathrm{t}_0$$

under the assumptions $\int_{t_0}^{\infty}\frac{d\eta}{a(\eta)}=\infty$, when H is sublinear and superlinear. Further, two illustrating examples are presented to show that feasibility and effectiveness of main results. Also, an open problem is included.

Keywords

Oscillation, nonlinear, sublinear, superlinear, delay, Lebesgue's Dominated Convergence theorem.

Mathematics Subject Classification 2010: 34C10, 34C15, 34K11.

1. Introduction

Consider a class of second order nonlinear delay differential equation of the form

$$\frac{d}{dt}\left[a(t)\frac{d}{dt}x(t)\right]+q(t)H\big(x(t-\sigma)\big)=0, \tag{1.1}$$

where

$$\sigma\in\mathbb{R}_+=(0,+\infty),\ q,a\in C(\mathbb{R}_+,\mathbb{R}_+),$$

and H is nondecreasing with

$$H\in C(\mathbb{R},\mathbb{R})\ \text{ with }\ uH(u)>0\ \text{ for }\ u\neq 0.$$

The main objective of this work is to establish necessary and sufficient conditions for oscillations of (1.1), under the assumptions

$$\int_{t_0}^{\infty}\frac{d\eta}{a(\eta)}=\infty$$

In [12], Santra has considered first-order nonlinear neutral delay differential equations of the form

$$\frac{d}{dt}\Big[x(t)+p(t)x(t-\tau)\Big]+q(t)H\big(x(t-\sigma)\big)=f(t) \tag{E$_1$}$$

And

$$\frac{d}{dt}\Big[x(t)+p(t)x(t-\tau)\Big]+q(t)H\big(x(t-\sigma)\big)=0. \tag{E$_2$}$$

He has studied oscillatory behaviour of the solutions of (E_1) and (E_2) under various ranges of p(t). Also, sufficient conditions are obtained for existence of bounded positive solutions of (E_1). In [13], Santra and Pinelas have established sufficient conditions for oscillatory and asymptotic behaviour of solutions of second order nonlinear neutral delay differential equations of the form

$$\frac{d}{dt}\left[a(t)\frac{d}{dt}[x(t)+p(t)x(t-\tau)]\right]+q(t)H\big(x(t-\sigma)\big)=0,\ \ t\geq t_0, \tag{E$_3$}$$

when $|p(t)|<+\infty$. The motivation of the present work has come from the above studies. We refer to the readers some related works [2], [3], [5], [7]- [9], [11], [14]- [18]. All of them are studied sufficient conditions for oscillation of second order non-linear functional differential equations only. But, necessary and sufficient conditions for oscillations of second order functional differential equations are much less attention. We may note that, this type of work is very rare in the literature.

An increasing interest in oscillation of solutions to functional differential equations during the last few decades has been stimulated by applications arising in engineering and natural sciences. The challenges that the new classes of such equations provide in these application areas. Equations involving delay, and those involving advance and a combination of both arise in the models on lossless transmission lines in high speed computers which are used to interconnect switching circuits. The construction of these models using delays is complemented by the mathematical investigation of nonlinear equations. Moreover, the delay differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons and to mention a few.

By a solution of (1.1) we understand a function $x\in C([-\sigma,\infty),\mathbb{R})$, such that x(t) and $a(t)x'(t)$ are once continuously differentiable and equation (1.1) is satisfied for t $\geq$0, where $\{|x(t)|:t\geq t_0\}>0$ for every t_0>0. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. Main Results

This section deals with the necessary and sufficient conditions for oscillation of all solutions of nonlinear second order delay differential equations of the form (1.1).

2.1 Hypotheses and statements of main results

We need the following notation and hypotheses on a(t) and H(u):

$$A(t)=\int_{t_0}^{t}\frac{d\eta}{a(\eta)}\quad\text{and}\quad\lim_{t\to\infty}A(t)=+\infty, \tag{A$_1$}$$

$$H(uv)=H(u)H(v),\ u,v\in\mathbb{R}. \tag{A$_2$}$$

For instance, the functions $a(t)=e^{-t}$ and $H(u)=u^{\gamma},\gamma\in\mathbb{R}$, the hypotheses (A1) and (A2) respectively.

Remark 2.1. [12] Assumption (A2) and definition of H implies that H is odd. Indeed, H(1)H(1) = H(1) and H(1) > 0 imply that H(1) = 1. Further, H(-1)H(-1) = H(1) = 1 implies that $(H(-1))^2 = 1$. Since H(-1) < 0, we conclude that H(-1) = -1. Hence,

$$H(-u)=H(-1)H(u)=-H(u)$$

On the other hand, H(uv) = H(u)H(v) for u > 0 and v > 0 and H(-u) = -H(u) imply that H(xy) = H(x)H(y) for every x; y $\in\mathbb{R}$.

Remark 2.2. [12] We may note that if x(t) is a solution of (1.1), then -x(t) is also a solution of (1.1) provided that H satisfies (A2).

We firstly consider the sublinear case of H(u), that is,

$$\frac{H(u)}{u^{\beta}}\geq\frac{H(v)}{v^{\beta}},\ 0<u\leq v,\ \beta<1. \tag{A$_3$}$$

Theorem 2.1. Assume that (A1), (A2) and (A3) hold. Then every solution of the equation (1.1) oscillates if and only if

$$\int_{\sigma}^{\infty}q(\eta)H\big(\varepsilon A(\eta-\sigma)\big)d\eta=+\infty\ for\ every\ \varepsilon>0. \tag{A$_4$}$$

Next, we consider the superlinear case of H(u), that is,

$$\frac{H(u)}{u^{\beta}}\geq\frac{H(v)}{v^{\beta}},\ u\geq v>0,\ \beta>1. \tag{A$_5$}$$

Theorem 2.2. Assume that (A1), (A2), (A5) hold and r(t) $\geq$ r(t –σ). Then every solution of the equation (1.1) is oscillatory if and only if

$$\int_{0}^{\infty}\frac{1}{a(\eta)}\left[\int_{\eta}^{\infty}q(\zeta)d\zeta\right]d\eta=+\infty. \tag{A$_6$}$$

2.2 Proofs of main results

Proof of Theorem 2.1. Let x(t) be a nonoscillatory solution of equation (1.1). So, there exists $t_0 > 0$ such that x(t) > 0 or < 0 for $t \geq t_0$. Without loss of generality and because of (A_2), we may assume that x(t) > 0 and $x(t-\sigma) > 0$ for $t \geq t_1 > t_0 + \sigma$. From (1.1), it follows that

$$\big(a(t)x'(t)\big)' = -q(t)H\big(x(t-\sigma)\big) < 0$$

for $t \geq t_1$. Hence there exists $t_2 > t_1$ such that $a(t)x'(t)$ is nonincreasing on $[t_2, \infty)$. We claim that $a(t)x'(t) > 0$ for $t \in [t_2, \infty)$. If $a(t)x'(t) \leq 0$ for $t \geq t_3$, then we can find c > 0 such that $a(t)x'(t) \leq -c$ for $t \geq t_3$. Integrating the relation $x'(t) \leq -\frac{c}{a(t)}$, $t \geq t_3$ from t_3 to t (> t_3) and obtain

$$x(t) \leq x(t_3) - c\int_{t_3}^{t} \frac{d\eta}{a(\eta)} \to -\infty, \;\; as \;\; t \to \infty.$$

due to (A_1), a contradiction to the fact that x(t) is a positive solution of the equation of (1.1). So, our claim holds. We integrate (1.1) from t ($\geq t_3$) to $+\infty$, we get

$$\big[a(\eta)x'(\eta)\big]_t^{\infty} + \int_t^{\infty} q(\eta)H\big(x(\eta-\sigma)\big)d\eta = 0. \tag{2.1}$$

Since, $a(t)x'(t)$ is nonincreasing on $[t_3, \infty)$, then there exists a constant c > 0 and $t_4 \geq t_3$ such that $a(t)x'(t) \leq c$ for $t \geq t_4$ and hence $x(t) \leq cA(t)$, where $t \geq t_4$. Using the fact H is sublinear, we have

$$\begin{aligned} H\big(x(t-\sigma)\big) &= \frac{H\big(x(t-\sigma)\big)}{x^{\beta}(t-\sigma)}x^{\beta}(t-\sigma) \\ &\geq \frac{H\big(cA(t-\sigma)\big)}{c^{\beta}A^{\beta}(t-\sigma)}x^{\beta}(t-\sigma), \end{aligned}$$

and hence (2.1) reduces to

$$\big[a(\eta)x'(\eta)\big]_t^{\infty} + \frac{1}{c^{\beta}}\int_t^{\infty} q(\eta)H\big(cA(\eta-\sigma)\big)\frac{x^{\beta}(\eta-\sigma)}{A^{\beta}(\eta-\sigma)}d\eta \leq 0. \tag{2.2}$$

Since, $a(t)x'(t)$ is decreasing function in $[t_4, \infty)$, then it follows that there exists the finite limit $\lim_{t\to\infty} a(t)x'(t) = B$, $B \in [0, \infty)$. Therefore, (2.2) becomes

$$\frac{1}{c^{\beta}}\int_t^{\infty} q(\eta)H\big(cA(\eta-\sigma)\big)\frac{x^{\beta}(\eta-\sigma)}{A^{\beta}(\eta-\sigma)}d\eta \leq a(t)x'(t)$$

for $t \geq t_4$, and hence

$$x'(t) \geq \frac{1}{c^{\beta}a(t)}\left[\int_t^{\infty} q(\eta)H\big(cA(\eta-\sigma)\big)\frac{x^{\beta}(\eta-\sigma)}{A^{\beta}(\eta-\sigma)}d\eta\right] \tag{2.3}$$

for $t \geq t_4$. Let $t_5 > t_4$ be such a point that

$$A(t) - A(t_5) \geq \frac{1}{2}A(t) \;\; for \;\; t \geq t_5.$$

Integrating (2.3) from t_5 to t (> t_5), we obtain

$$\begin{aligned} x(t) - x(t_5) &\geq \frac{1}{c^{\beta}}\int_{t_5}^{t} \frac{1}{a(\eta)}\left[\int_{\eta}^{\infty} q(\zeta)H\big(cA(\zeta-\sigma)\big)\frac{x^{\beta}(\zeta-\sigma)}{A^{\beta}(\zeta-\sigma)}d\zeta\right]d\eta \\ &\geq \frac{1}{c^{\beta}}\int_{t_5}^{t} \frac{1}{a(\eta)}\left[\int_{t}^{\infty} q(\zeta)H\big(cA(\zeta-\sigma)\big)\frac{x^{\beta}(\zeta-\sigma)}{A^{\beta}(\zeta-\sigma)}d\zeta\right]d\eta \end{aligned}$$

that is,

$$\begin{aligned} x(t) &\geq \frac{1}{c^{\beta}}\int_{t_5}^{t} \frac{1}{a(\eta)}\left[\int_{t}^{\infty} q(\zeta)H\big(cA(\zeta-\sigma)\big)\frac{x^{\beta}(\zeta-\sigma)}{A^{\beta}(\zeta-\sigma)}d\zeta\right]d\eta \\ &\geq \frac{1}{2c^{\beta}}A(t)\left[\int_{t}^{\infty} q(\zeta)H\big(cA(\zeta-\sigma)\big)\frac{x^{\beta}(\zeta-\sigma)}{A^{\beta}(\zeta-\sigma)}d\zeta\right] \end{aligned}$$

for $t \geq t_5$. If we define

$$w(t) = \frac{1}{2c^\beta}\left[\int_t^\infty q(\zeta)H\big(cA(\zeta-\sigma)\big)\frac{x^\beta(\zeta-\sigma)}{A^\beta(\zeta-\sigma)}d\zeta\right],$$

then $x(t) \geq A(t)w(t)$ for $t \geq t_5$. Now,

$$\begin{aligned} w'(t) &= -\frac{1}{2c^\beta}q(t)H\big(cA(t-\sigma)\big)\frac{x^\beta(t-\sigma)}{A^\beta(t-\sigma)} \\ &\leq -\frac{1}{2c^\beta}q(t)H\big(cA(t-\sigma)\big)w^\beta(t-\sigma) \leq 0 \end{aligned}$$

implies that $w(t)$ is nonincreasing on $[t_5, \infty)$ and $\lim_{t\to\infty} w(t)$ exists. It is easy to verify that

$$\begin{aligned} \left[w^{1-\beta}(t)\right]' &\leq -\frac{1}{2c^\beta}(1-\beta)q(t)H\big(cA(t-\sigma)\big)w^{-\beta}(t)w^\beta(t-\sigma) \\ &\leq -\frac{1}{2c^\beta}(1-\beta)q(t)H\big(cA(t-\sigma)\big). \end{aligned}$$

Integrating the last inequality from t_5 to $t(> t_5)$, we obtain

$$\left[w^{1-\beta}(\eta)\right]_{t_5}^t \leq -\frac{1}{2}(1-\beta)c^{-\beta}\int_{t_5}^t q(\eta)H\big(cA(\eta-\sigma)\big)d\eta.$$

that is,

$$\begin{aligned} \frac{1}{2}(1-\beta)c^{-\beta}\int_{t_5}^t q(\eta)H\big(cA(\eta-\sigma)\big)d\eta &\leq -\left[w^{1-\beta}(\eta)\right]_{t_5}^t \\ &< \infty, \quad as \ t\to\infty, \end{aligned}$$

a contradiction to (A_4).

Next, we suppose that (A_4) doesn't hold. So, for $T > \sigma$ and $c > 0$, let

$$\int_T^\infty q(\eta)H\big(cA(\eta-\sigma)\big)d\eta < \frac{c}{2}.$$

Let's consider

$$\begin{aligned} M = \{x : x \in C([T-\sigma,+\infty),\mathbb{R}), x(t) = 0 \ for\ t \in [T-\sigma, T]\ and \\ \frac{c}{2}[A(t)-A(T)] \leq x(t) \leq c[A(t)-A(T)]\} \end{aligned}$$

and define $\Omega: M \to C([T-\sigma, +\infty), \mathbb{R})$ such that

$$(\Omega x)(t) = \begin{cases} 0, & t \in [T-\sigma, T) \\ \int_T^t \frac{1}{a(\eta)}\left[\frac{c}{2} + \int_\eta^\infty q(\zeta)H\big(x(\zeta-\sigma)\big)d\zeta\right]d\eta, & t \geq T. \end{cases}$$

For every $x \in M$,

$$(\Omega x)(t) \geq \frac{c}{2}\int_T^t \frac{d\eta}{a(\eta)} = \frac{c}{2}\left[A(t)-A(T)\right],$$

and the inequality $x(t) \leq cA(t)$ implies that

$$(\Omega x)(t) \leq c\int_T^t \frac{d\eta}{a(\eta)} = c\left[A(t)-A(T)\right].$$

Thus, $(\Omega x)(t) \in M$. Let us define now the function $u_n: [T-\sigma, +\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = (\Omega u_{n-1})(t), \quad n \geq 1,$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [T-\sigma, T) \\ \frac{c}{2}[A(t)-A(T)], & t \geq T. \end{cases}$$

Inductively it is easily verified that

$$\frac{c}{2}[A(t)-A(T)] \le u_{n-1}(t) \le u_n(t) \le c[A(t)-A(T)]$$

for $t \geq T$: Therefore for $t \geq T-\sigma$, $\lim_{n\to+\infty} u_n(t)$ exists. Let $\lim_{n\to+\infty} u_n(t) = u(t)$ for $t \geq T-\sigma$. By Lebesgue's Dominated Convergence theorem $u \in M$ and $(u)(t) = u(t)$, where $u(t)$ is a solution of the equation (1.1) on $[T-\sigma, \infty)$ such that $u(t) > 0$. Hence, (A4) is a necessary condition. This completes the proof of the theorem.

Proof of Theorem 2.2. For sufficient part, we use the same type of argument as in the proof of the Theorem 2.1 for the case $a(t)\,x'(t) \leq 0$. Let's consider the case $a(t)\,x'(t) > 0$ for $t \geq t_3$. So there exists a constant $c > 0$ and $t_4 > t_3$ such that $x(t-\sigma) \geq c$ for $t \geq t_4$. Consequently,

$$\begin{aligned} H\big(x(t-\sigma)\big) &= \frac{H\big(x(t-\sigma)\big)}{x^\beta(t-\sigma)} x^\beta(t-\sigma) \\ &\ge \frac{H(c)}{c^\beta} x^\beta(t-\sigma),\ t \ge t_4, \end{aligned}$$

due to (A5). Therefore, (2.1) becomes

$$\big[a(\eta)x'(\eta)\big]_t^\infty + \int_t^\infty q(\eta)\frac{H(c)}{c^\beta} x^\beta(t-\sigma)d\eta \le 0. \tag{2.4}$$

From the fact that $a(t)\,x'(t)$ is decreasing function in $[t_4, \infty)$, it follows that there exists the finite limit $\lim_{t\to\infty} a(t)x'(t) = B$, $B \in [0, \infty)$. Therefore, (2.3) becomes

$$a(t)x'(t) \ge \int_t^\infty q(\eta)\frac{H(c)}{c^\beta} x^\beta(\eta-\sigma)d\eta,$$

that is,

$$a(t-\sigma)x'(t-\sigma) \ge \frac{H(c)}{c^\beta}\left[\int_t^\infty q(\eta)x^\beta(\eta-\sigma)d\eta\right],$$

for $t > t_4$ implies that

$$\begin{aligned} x'(t-\sigma) &\ge \frac{H(c)}{c^\beta a(t-\sigma)}\left[\int_t^\infty q(\eta)d\eta\right] x^\beta(t-\sigma) \\ &\ge \frac{H(c)}{c^\beta r(t)}\left[\int_t^\infty q(\eta)d\eta\right] x^\beta(t-\sigma). \end{aligned}$$

Integrating the last inequality from t_4 to $+\infty$, we get

$$\frac{H(c)}{c^\beta}\int_{t_4}^\infty \frac{1}{a(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \le \int_{t_4}^\infty \frac{x'(\eta-\sigma)}{x^\beta(\eta-\sigma)} \le \frac{x^{1-\beta}(t_4-\sigma)}{\beta-1} < +\infty,$$

which is a contradiction to (A6).

Next, we show that (A6) is necessary. Assume that (A6) fails to hold and let

$$H(c)\int_T^\infty \frac{1}{a(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \le \frac{c}{2},\ \ T \ge \sigma, \tag{2.5}$$

where $c > 0$ is a constant. Consider

$$\begin{aligned} M = \{x : x \in C([T-\sigma, +\infty), \mathbb{R}),\ x(t) = \frac{c}{2}\ for\ t \in [T-\sigma, T)\ and \\ \frac{c}{2} \le x(t) \le c,\ t \ge T\}, \end{aligned}$$

and let $\Omega: M \to C([T-\sigma, +\infty), \mathbb{R})$ be defined by

$$(\Omega x)(t) = \begin{cases} \frac{c}{2}, & t \in [T-\sigma, T) \\ \frac{c}{2} + \int_T^t \frac{1}{a(\eta)}\left[\int_\eta^\infty q(\zeta)H\big(x(\zeta-\sigma)\big)d\zeta\right]d\eta, & t \ge T. \end{cases}$$

For every $x \in M$, $(\Omega x)(t) \ge \frac{c}{2}$, Using definition of the set M, definition of the mapping Ω and (2.5), we obtained $(\Omega x)(t) < c$. Therefore, $(\Omega x) \le M$. Analogously to the proof of the Theorem 2.1 we get that the mapping Ω has a fixed point $u \in M$, that is, $u(t) = (\Omega u)(t)$, $t \geq T-\sigma$. It can be easily verified that $u(t)$ is a solution of (1.1), such that $\frac{c}{2} \le u(t) \le c$

for t $\geq T$, that is, u(t) is a nonoscillatory solution of (1.1). Thus the proof of the theorem is complete.

3. Examples

We conclude this paper with the following examples to illustrate our main results:
Example 3.1. Consider the delay differential equations

$$\frac{d}{dt}\left[e^{-t}\frac{d}{dt}x(t)\right]+e^{t}\left(x(t-2)\right)^{\frac{1}{3}}=0, \tag{E$_4$}$$

where $a(t) = e^{-t}$, $q(t) = e^{t}$, $\sigma = 2$ and $H(x) = x^{\frac{1}{3}}$. If we choose $\beta = \frac{1}{2} < 1$, then all the assumptions of the Theorem 2.1 holds. Hence by Theorem 2.1, every solution of (E_4) oscillates.
Example 3.2. Consider the delay differential equations

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}x(t)\right]+e^{2t}\left(x(t-1)\right)^{3}=0, \tag{E$_5$}$$

where $q(t) = e^{2t}$, $\sigma = 1 = a(t)$ and $H(x) = x^3$. If we choose $\beta = 2 > 1$, then all the assumptions of the Theorem 2.2 holds. Hence by Theorem 2.2, every solution of (E_5) oscillates.

Open Problem

Can we find necessary and sufficient condition for oscillation and asymptotic behaviour of second order nonlinear neutral delay differential equations of the form (E_3) for $|p(t)| < +\infty$ and using the assumption (A_1)?

Competing interests

The author declares that they have no competing interests.

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References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation Theory for Second Order Dynamic Equations, Taylor & Francis, London and New York, 2003.
[2] B. Baculikova, T. Li, J. Dzurina; Oscillation theorems for second order neutral differential equations, Elect. J. Quali. Theo. diff. equa., (74): (2011), 1-13.
[3] J. Dzurina; Oscillation theorems for second order advanced neutral differential equations, Tatra Mt. Math. Publ., DOI: 10.2478/v10127-011-0006-4, 48 (2011), 61-71.
[4] I. Gyori, G. Ladas; Oscillation Theory of Delay Differential Equations with Applications, Clarendon, Oxford, 1991.
[5] M. Hasanbulli, Y. V. Rogovchenko; Oscillation criteria forsecond order nonlinear neutral differential equations, Appl. Math. Comput., 215 (2010), 4392-4399.
[6] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York and Basel, 1987.
[7] T. Li, Yu. V. Rogovchenko, C. Zhang; Oscillation results for second-order nonlinear neutral differential equations, Adv. Di_erence Equ., 2013 (2013), 1-13.
[8] T. Li, Yu. V. Rogovchenko; Oscillation theorems for second-order nonlinear neutral delay differential equations, Abst. Appl. Anal, 2014 (2014), 1-5.
[9] Q. Li, R. Wang, F. Chen, T. Li; Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients, Adv. Diff. Eq. (2015) 2015:35. DOI 10.1186/s13662-015-0377-y.
[10] Y. Liu, J. Zhanga, J. Yan; Existence of oscillatory solutions of second order delay differential equations, J. Comp. Appl. Math., 277 (2015), 17-22.
[11] Q. Meng, J. Yan; Bounded oscillation for second order non-linear neutral delay differential equations in critical and non-critical cases, Nonlinear Anal., 64 (2006), 1543-1561.
[12] S. S. Santra; Existence of positive solution and new oscillation criteria for nonlinear first-order neutral delay differential equations, Diff. Equ. Appl., 8(1): (2016), 33-51.

[13] S. S. Santra, S. Pinelas; Qualitative behaviour for second order nonlinear delay differential equations of neutral type, Global J. Math., 8(3): (2016), 939-956.
[14] S. Tanaka; A oscillation theorem for a class of even order neutral differential equations, J. Math. Anal. Appl., 273 (2007), 172-189.
[15] R. Xu, F. Meng; Some new oscillation criteria for second order quasilinear neutral delay differential equations, Appl. Math. Comp., 182 (2006), 797-803.
[16] Z. Xu, P. Weng; Oscillation of second order neutral equations with distributed deviating argument, J. Comp. Appl. Math., 202 (2007), 460-477.
[17] J. Yan; Existence of oscillatory solutions of forced second order delay differential equations, Appl. Math. Lett. 24 (2011), 1455-1460.
[18] Q. Zhang, J. Yan; Oscillation behavior of even order neutral differential equations with variable coefficients, Appl. Math. Lett., 19 (2006), 1202-1206.