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The Pell, Modified Pell Identities Via Orthogonal Projection

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Abstract

In this paper, we consider the space $\mathrm{R}(2,1)$ of modified Pell sequences and orthogonal bases of this space. These orthogonal bases are connected with the modified Pell numbers and Pell numbers. Using this orthogonal bases, we obtain the orthogonal projection onto a subspace $\mathrm{R}(2,1)$ of $\mathbb{R}^n$. By using the orthogonal projection, we obtain the identities for the Pell, Pell-Lucas and modified Pell numbers.

Keywords

Orthogonal projection, Pell numbers, Binet's formula.

1. Introduction

Firstly, we shall give the definitions and some properties of Pell, Pell-Lucas and modified Pell numbers.
The Pell numbers P_n are defined by following recurrence relation for $n \geq 1$

$$P_{n+1} = 2P_n + P_{n-1} \tag{1.1}$$

with the initial conditions $P_0 = 0, P_1 = 1$. Similarly, the modified Pell numbers q_n are defined by Horadam as the following recurrence relation

$$q_{n+1} = 2q_n + q_{n-1} \tag{1.2}$$

where $q_0 = q_1 = 1$. The recurrence relation of the Pell-Lucas numbers Q_n are defined as

$$Q_{n+1} = 2Q_n + Q_{n-1} \tag{1.3}$$

with the initial conditions $Q_0 = 2, Q_1 = 2$.
The characteristic equation of recurrences (1.1), (1.2) and (1.3) is

$$\lambda^2 - 2\lambda - 1 = 0 \tag{1.4}$$

The Binet's formula for Pell, Pell-Lucas and modified Pell numbers are

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ Q_n = \alpha^n + \beta^n \ \text{ and } \ q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}$$

where α and β are roots of the equation (1.4).
Now, we give the theorems about orthogonal projections.

Theorem 1.1 ([2], p. 204, Theorem 5.1.4). Consider a vector $\vec{x} \in \mathbb{R}^n$ and subspace V of $\mathbb{R}^n$. Then we can write $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ where $\vec{x}^{\parallel}$ is in V and $\vec{x}^{\perp}$ is perpendicular to V and this representation is unique. The vector

$\overrightarrow{x}^{\shortparallel}$ is called the orthogonal projection of $\overrightarrow{x}$ onto V, denoted by $proj_V(\overrightarrow{x})$. The transformation $T(\overrightarrow{x}) = proj_V(\overrightarrow{x}) = \overrightarrow{x}^{\shortparallel}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is linear.

Theorem 1.2 ([2], p. 206, Theorem 5.1.5). If V is a subspace of $\mathbb{R}^n$ with an orthonormal bases $u_1, u_2, \ldots, u_m$, then

$$proj_V(x) = \langle u_1, x\rangle u_1 + \cdots + \langle u_m, x\rangle u_m \tag{1.5}$$

for all x in $\mathbb{R}^n$.

By using the equation (1.5), we can give the matrix of orthogonal projection as the following theorem.

Theorem 1.3 ([2], p. 232, Theorem 5.3.10). Consider a subspace V of $\mathbb{R}^n$ with orthonormal bases $u_1, u_2, \ldots, u_m$. The matrix P of the orthogonal projection onto V is

$$P = u_1 u_1^T + \cdots + u_m u_m^T.$$

Theorem 1.4 ([4], p. 365, Theorem 6.12). The projection matrix P for subspace V of $\mathbb{R}^n$ is both idempotent and symmetric. Conversely, every $n \times n$ matrix that is both idempotent and symmetric is a projection matrix.

Many authors have investigated on the second order recurrence sequences. Specially, we consider studies connected with the Pell, Pell-Lucas and modified Pell numbers. In [7], the author defines the modified Pell numbers and give properties of the modified Pell numbers. In [1], the author consider the Toeplitz and Hankel matrices with the Pell, Pell-Lucas and modified Pell numbers and investigate norms, determinants of these matrices. Halıcı investigate the some formulae for the Pell, Pell-Lucas and modified Pell numbers in [5]. In [8], the authors consider the general Fibonacci numbers and give the interesting properties. Dupree and Mathes investigate the singular values of Hankel matrices with $k-$Fibonacci and $k-$Lucas numbers. Also, they give the orthogonal projection onto the two dimensional space of $k-$Fibonacci and $k-$Lucas sequences in [3]. In [6], the authors consider the orthogonal projection onto the two dimensional space of $k-$Fibonacci and $k-$Lucas sequences in [3] and give a new proof of obtained results by Dupree and Mathes.

Motivated by the above papers, we investigate the orthogonal projection onto the two dimensional space of modified Pell sequences. We can see that the obtained orthogonal projection matrix is Hankel matrix with the Pell numbers entries. Also, we give the identities for the Pell, Pell-Lucas and modified Pell numbers by using the orthogonal projection matrix.

2. MAIN RESULTS

Let $\mathbb{R}^n$ denote the n dimensional vector space of all real $n-$tuples, and let $\mathrm{R}(2,1)$ denote the subspace of $\mathbb{R}^n$, consisting of the $q_i \in \mathbb{R}^n$
which

$$q_{i+1} = 2q_i + q_{i-1}$$

for $\mathrm{i} = 1, 2, \ldots, n$. The element of $\mathrm{R}(2,1)$ whose first two coordinates 1 and 1 will be denoted (q_i) and is called the modified Pell sequence.

In this paper, we obtain the matrix of orthogonal projection onto $\mathrm{R}(2,1)$ as follows

$$\frac{2}{P_n}\begin{pmatrix} P_{-n+1} & P_{-n+2} & \cdots & P_0 \\ P_{-n+2} & P_{-n+3} & \cdots & P_1 \\ \vdots & \vdots & \ddots & \vdots \\ P_0 & P_1 & \cdots & P_{n-1} \end{pmatrix} \tag{2.1}$$

for n is even. It's note that, the matrix

$$H_p = \begin{pmatrix} P_{-n+1} & P_{-n+2} & \cdots & P_0 \\ P_{-n+2} & P_{-n+3} & \cdots & P_1 \\ \vdots & \vdots & \ddots & \vdots \\ P_0 & P_1 & \cdots & P_{n-1} \end{pmatrix}$$

is called the central Hankel matrix with the Pell numbers entries. Replacing Pell numbers with Pell-Lucas numbers yields H_Q central Pell-Lucas Hankel matrix. For n is odd, we obtain the matrix of orthogonal projection onto $\mathrm{R}(2,1)$ which is connected with the central Pell-Lucas Hankel matrix H_Q.

Let E denote the matrix of order 2

$$E=\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}$$

It follows that

$$E^m=\begin{pmatrix}P_{m-1} & P_m\\ P_m & P_{m+1}\end{pmatrix} \tag{2.2}$$

for all integer m.

Lemma 2.1. For any integer t and any nonnegative integer s, we have

$$\sum_{i=0}^{s}E^{t+4i}=\frac{P_{2(s+1)}}{2}E^{t+2s} \quad . \tag{2.3}$$

Proof . It suffices to prove the identity

$$\sum_{i=0}^{s}P_{t+4i}=\frac{P_{2(s+1)}}{2}P_{t+2s}.$$

Using the Binet's formula for the Pell numbers, we have

$$\begin{aligned}\sum_{i=0}^{s}P_{t+4i}&=\sum_{i=0}^{s}\frac{\alpha^{t+4i}-\beta^{t+4i}}{\alpha-\beta}\\ &=\frac{\alpha^t}{\alpha-\beta}\sum_{i=0}^{s}\alpha^{4i}-\frac{\beta^t}{\alpha-\beta}\sum_{i=0}^{s}\beta^{4i}\\ &=\frac{\left(\alpha^{4s+t}-\beta^{4s+t}\right)-\left(\alpha^{4s+t+4}-\beta^{4s+t+4}\right)-\left(\alpha^{t-4}-\beta^{t-4}\right)+\left(\alpha^{t}-\beta^{t}\right)}{(\alpha-\beta)\left(\alpha^4-1\right)\left(\beta^4-1\right)}\\ &=-\frac{1}{32}\left(P_{4s+t}-P_{4s+t+4}-P_{t-4}+P_t\right).\end{aligned}$$

From the identity $2Q_{n-2}=P_n-P_{n-4}$, we obtain

$$\sum_{i=0}^{s}P_{t+4i}=\frac{1}{16}(Q_{4s+t+2}-Q_{t-2}).$$

In [5], the author gives the following identity

$$P_nP_{n+k}=\frac{1}{8}(Q_{2n+k}+(-1)^{n+1}Q_k) \quad .$$

Taking $n=2s+2$ and $k=t-2$ in the above identity, we have

$$\begin{aligned}\sum_{i=0}^{s}P_{t+4i}&=-\frac{1}{32}\left(2Q_{t-2}-2Q_{4s+t+2}\right)\\ &=\frac{P_{2(s+1)}}{2}P_{2s+t} \quad .\end{aligned}$$

Theorem 2.1. For even n, the matrix $\frac{2}{P_n}H_p$ is orthogonal projection matrix onto $\mathbb{R}(2,1)$.

Proof. From Theorem 1.4, orthogonal projection matrix is both symmetric and idempotent. The matrix $\frac{2}{P_n}H_p$ is clearly symmetric. Therefore , we need show only that $\frac{2}{P_n}H_p$ is idempotent. Namely, we will prove that

$$H_p^2 = \frac{P_n}{2}H_p.$$

We can express the matrix H_p as

$$H_p = \begin{pmatrix} E^{-n+2} & E^{-n+4} & \dots & E^{-2} & E^{0} \\ E^{-n+4} & E^{-n+6} & \dots & E^{0} & E^{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ E^{-2} & E^{0} & \dots & E^{n-6} & E^{n-4} \\ E^{0} & E^{2} & \dots & E^{n-4} & E^{n-2} \end{pmatrix}$$

with the matrix E^m. Taking $n=2m$, we have

$$H_p = \left[E^{2(i+j)-2(m+1)}\right]_{i,j=1}^{m}.$$

Using the Lemma 2.1, we obtain

$$\begin{aligned} H_p^2 &= \left[\sum_{r=1}^{m} E^{2(i+r)-2(m+1)}E^{2(r+j)-2(m+1)}\right]_{i,j=1}^{m} \\ &= \left[\sum_{r=1}^{m} E^{2(i+j)-4m+4(r-1)}\right]_{i,j=1}^{m} \\ &= \frac{P_{2m}}{2}\left[E^{2(i+j)-4m+2(m-1)}\right]_{i,j=1}^{m} \\ &= \frac{P_n}{2}\left[E^{2(i+j)-2(m+1)}\right]_{i,j=1}^{m} \\ &= \frac{P_n}{2}H_p. \end{aligned}$$

Using the idempotency of the matrix $\frac{2}{P_n}H_p$, we obtain the identity for the Pell numbers as

$$\frac{2}{P_n}P_{i+j-n+1} = \frac{4}{P_n^2}\sum_{k=0}^{n-1} P_{i+k-n+1}P_{j+k-n+1}$$

for all even n and $-n+1 \le i, j \le n-1$. Thus, we give the following corollary.

Corollary 2.1. For even n and $-n+1 \le i, j \le n-1$, we have

$$P_n P_{i+j-n+1} = 2\sum_{u=0}^{n-1} P_{i-u}P_{j-u}\ .$$

Assume that n is even and $s = (q_0, q_1, \dots, q_{n-2}, q_{n-1}) \in \mathbb{R}(2,1)$ which is a column vector. We define

$$s^{\perp} = (-q_{n-1}, q_{n-2}, \dots, -q_1, q_0)^T.$$

It's clear that $\{s, s^{\perp}\}$ is orthogonal bases for the space $\mathrm{R}(2,1)$. Normalizing s and $s^{\perp}$, we consider the u and v vectors as

$$u = \frac{s}{\|s\|}, v = \frac{s^{\perp}}{\|s^{\perp}\|}$$

where

$$\|s\|^2 = \|s^{\perp}\|^2 = \frac{q_{2n-1}+1}{4}.$$

From Theorem 1.3, the matrix of orthogonal projection onto $\mathrm{R}(2,1)$ is

$$P = uu^T + vv^T \tag{2.4}$$

which is the Hankel matrix in (2.1).

Theorem 2.2. For even n and $0 \le i, j \le n-1$, we have

$$2P_{i+j-n+1}P_{n-1} = q_i q_j + (-1)^{i+j} q_{n-i-1}q_{n-j-1} \tag{2.5}$$

Proof. By equalizing the $ij-th$ entries of the matrices in (2.1) and (2.4), we have

$$\frac{4}{q_{2n-1}+1}\left(q_i q_j + (-1)^{i+j} q_{n-i-1}q_{n-j-1}\right) = \frac{2}{P_n} P_{i+j-n+1}$$

$$2P_n\left(q_i q_j + (-1)^{i+j} q_{n-i-1}q_{n-j-1}\right) = P_{i+j-n+1}\left(q_{2n-1}+1\right)$$

Using the following identity, we simplify the above equation

$$P_m P_{m+1} = \frac{1}{4}\left(q_{2m+1} + (-1)^{m+1}\right).$$

Taking $m = n-1$ and even n, we obtain

$$2P_{i+j-n+1}P_{n-1} = q_i q_j + (-1)^{i+j} q_{n-i-1}q_{n-j-1}$$

By using the above theorem, we have the identities for the Pell and modified Pell numbers as follows.

For $i = j = 0$ and $i = j = n-1$ in (2.5), we obtain

$$2P_{n-1}^2 = q_{n-1}^2 + 1.$$

Taking $i = j = \frac{n}{2}$ in (2.5), we have

$$2P_{n-1} = q_{\frac{n}{2}}^2 + q_{\frac{n}{2}-1}^2.$$

Let $i = n-1, j = 1$ in (2.5), then

$$2P_{n-1} = q_{n-1} + q_{n-2}.$$

Let α and β are roots of the characteristic equation (1.4). We consider the another orthogonal bases of $\mathrm{R}(2,1)$ which is $\{s, t\}$ where

$$s = (1, \alpha, \alpha^2, \ldots, \alpha^{n-1})^{\mathrm{T}} \text{ and } t = (1, \beta, \beta^2, \ldots, \beta^{n-1})^{\mathrm{T}}$$

The orthogonal projection matrix onto $\mathrm{R}(2,1)$ is

$$P = \frac{1}{\|s\|^2} ss^T + \frac{1}{\|t\|^2} tt^T. \tag{2.6}$$

This matrix is Hankel matrix with the Pell numbers in (2.1).

Theorem 2.3. For n is even, we have

$$\frac{\alpha^{i+j+1}}{\alpha^{2n}-1}+\frac{\beta^{i+j+1}}{\beta^{2n}-1}=\frac{1}{P_n}P_{i+j-n+1} \tag{2.7}$$

for all $0\le i, j\le n-1$.

Proof. By equalizing the $ij-th$ entries of the matrices in (2.1) and (2.6), we have

$$\frac{\alpha^2-1}{\alpha^{2n}-1}\alpha^{i+j}+\frac{\beta^2-1}{\beta^{2n}-1}\beta^{i+j}=\frac{1}{P_n}P_{i+j-n+1}.$$

For $\alpha^2-1=2\alpha$ and $\beta^2-1=2\beta$, the result is clear.

Specially, taking $i=1$ and $j=n-1$ in (2.7), we have

$$\frac{\alpha^{n+1}}{\alpha^{2n}-1}+\frac{\beta^{n+1}}{\beta^{2n}-1}=\frac{1}{P_n}$$

Using $\alpha\beta=-1$ and n is even,

$$\left(\frac{\alpha^{n+1}}{\alpha^{2n}-1}+\frac{\beta^{n+1}}{\beta^{2n}-1}\right)\left(\alpha^n-\beta^n\right)=\left(\alpha-\beta\right)$$

Hence, the Binet's formula for Pell numbers appears as follows

$$P_n=\frac{\left(\alpha^n-\beta^n\right)}{\alpha-\beta}.$$

We consider odd values of n, $n=2m+1$. Now, let u and v modified Pell and Pell sequences which are

$$u=(q_{-m},\dots,q_0,\dots,q_m)^T \text{ and } v=(p_{-m},\dots,p_0,\dots,p_m)^T$$

respectively. These vectors are eigenvectors of the matrix H_Q. We have the norms of u and v

$$\|u\|^2=\frac{q_{2m+1}+(-1)^m}{2} \quad \text{and} \quad \|v\|^2=P_mP_{m+1}.$$

Also, using the fact that $q_{-i}=(-1)^i q_i$ and $P_{-i}=(-1)^{i+1}P_i$, we have

$$\sum_{i=-m}^{m}q_iP_i=-\sum_{i=1}^{m}q_iP_i+q_0P_0+\sum_{i=0}^{m}q_iP_i=0.$$

Namely $\{u,v\}$ is orthogonal base of $\mathrm{R}(2,1)$. The orthogonal projection matrix onto $\mathrm{R}(2,1)$ is given by

$$P=\frac{2}{q_{2m+1}+(-1)^m}uu^T+\frac{1}{P_mP_{m+1}}vv^T \tag{2.8}$$

The following theorem gives the second expression for this projection.

Theorem 2.4. The orthogonal projection matrix in *(2.8)* is

$$\frac{4(-1)^m}{P_{2m}P_{2m+2}}vv^T+\frac{1}{2q_mq_{m+1}}H_Q \tag{2.9}$$

Proof. Let consider the Hankel matrix with Pell-Lucas numbers entries, H_Q. The eigenvalues of H_Q are $2q_mq_{m+1}$ and $4P_mP_{m+1}$. The eigenvectors u and v are connected with this eigenvalues.

$$H_Q=2q_mq_{m+1}\frac{2}{q_{2m+1}+(-1)^m}uu^T+4P_mP_{m+1}\frac{1}{P_mP_{m+1}}vv^T$$

$$= 4q_m q_{m+1} \frac{1}{q_{2m+1} + (-1)^m} uu^T + 4vv^T \quad .$$

The projection matrix in (2.8) is

$$P = \frac{2}{q_{2m+1} + (-1)^m} uu^T + \frac{1}{P_m P_{m+1}} vv^T$$

$$= \frac{2}{q_{2m+1} + (-1)^m} \left[\frac{q_{2m+1} + (-1)^m}{4q_m q_{m+1}} \left(H_Q - 4vv^T \right) \right] + \frac{1}{P_m P_{m+1}} vv^T$$

$$= \left(\frac{1}{P_m P_{m+1}} - \frac{2}{q_m q_{m+1}} \right) vv^T + \frac{1}{2q_m q_{m+1}} H_Q$$

Using the following identity, we simplify the above equation

$$q_{2n+1} = 2P_n P_{n+1} + q_n q_{n+1}$$

Namely, we have

$$P = \frac{4(-1)^m}{P_{2m} P_{2m+2}} vv^T + \frac{1}{2q_m q_{m+1}} H_Q \quad .$$

Corollary 2.2. If n is odd, then

$$\frac{2q_i q_j}{q_{2m+1} + (-1)^m} + \frac{P_i P_j}{P_m P_{m+1}} = \frac{4(-1)^m P_i P_j}{P_{2m} P_{2m+2}} + \frac{Q_{i+j}}{2q_m q_{m+1}} \tag{2.10}$$

for $-m \le i, j \le m$.

Proof. By equalizing the $ij-th$ entries of the matrices in (2.8) and (2.9), the result is clear.

Taking $i = 1$ and $j = 0$ in (2.10), we have the identity for the modified Pell numbers as follows

$$q_{2m+1} = 2q_m q_{m+1} - (-1)^m .$$

Similarly we obtain

$$\frac{1}{P_{m+1}} - \frac{1}{q_m} = \frac{4(-1)^m}{Q_m P_{2m+2}} - \frac{2q_m}{q_{2m+1} + (-1)^m}$$

for $i = m$ and $j = 1$.

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