Journal of Applied Mathematics and Computation (JAMC), 2018, 2(3), 67-83
http://www.hillpublisher.org/journal/jamc
ISSN Online:2576-0645
ISSN Print:2576-0653

Analysis of two-dimensional solute transport through heterogeneous porous medium

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How to cite this paper: Kumar, A. et al., (2018) Analysis of two-dimensional solute transport through heterogeneous porous medium. *Journal of Applied Mathematics and Computation*, **2(3)**, 67-83.
http://dx.doi.org/10.26855/jamc.2018.03.001

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Abstract

In this study, an analytical solution is developed for solute transport through heterogeneous porous medium in two-dimensional horizontal Cartesian plane. In the present problem, the domain is considered semi-infinite extent and of heterogeneous nature along the both perpendicular directions of the plane. In surface water body, the aquifer of shallow depths and lateral component of velocity also considered. Due to heterogeneity of the medium, the velocity components of the flow transporting the solute through the medium along both the directions are considered spatially dependent. The velocity along each direction is supposed to increase linearly while passing through a finite region (into which the solute concentration values are evaluated) starting from the origin, by using interpolating formula. Solute dispersion along both the directions is considered proportional to square of the respective velocity component. The present problem is derived for two cases: first one is uniform pulse type input point source and other one is for varying pulse type input point source. The first input condition is considered initially and second is the far end of the domain. It is considered flux type of homogeneous nature. In the both cases, domain is considered initially not solute free. In the process of analytical solutions, a new space or positional variable is introduced. After that, introducing the Laplace transform to get the analytical solutions of our problems. In both the cases, effects of heterogeneity/inhomogeneity of the medium and solute/solute transport behavior have discussed in the presence and absence of the source. All the possible solute/solute transport behaviors are shown graphically interpreted through MatLab 7.0.

Keywords

Advection-Dispersion Equation, Heterogeneous/Inhomogeneous Medium, Pulse type input point source, Diffusion Processes, Laplace Transformation, Interpolation.

1. Introduction

Interpreting water quality and ecological implications in stream systems depends on accurate predictions of the fate and transport of solute and heat. Applying a representative solute and heat transport model that incorporates the influences of surface and hyporheic transient storage requires knowledge of individual, dominant processes over different spatial scales. However, estimating parameters that represent those dominant processes and determining appropriate residence time distributions are common challenges due to the inherent heterogeneity of characteristics within streams. Recent progress has been made to better represent residence times by scaling parameters with field-based geometric and hydraulic measurements. Despite this advancement, it is still unclear what spatial detail of observations is necessary and how best to represent that detail in reach scale model applications. Anticipating how stream water quality will respond to change, such as increased pollution or water diversions, requires knowledge of the main mechanisms controlling water and chemical constituent movement and a reasonable representation of those mechanisms. By deriving mathematical models to represent a stream system and collecting supporting field-based measurements, water quality response can be predicted. However, because each stream is unique and the movement of water and constituents is spatially and temporally complex, assessing whether the stream is appropriately represented and whether predictions are trustworthy is still a challenge within the scientific and management communities.The transport of contaminant through porous media is a very complex process and it is dominated by several physical and chemical factors related to porous medium. Any variations in the physical and chemical properties of the porous medium induce an imbalance in the transport of contaminant. Solute transport through porous media is mathematically modeled by means of the advection-dispersion equation. On such modeling is useful in recognizing and determining the extent of contamination in aquifers.

Groundwater pollution occurs due to infiltration of wastes through rainwater, from garbage disposal sites, septic tanks and mines, and discharge from surface water bodies polluted due to industrial and municipal influents. If groundwater becomes polluted it is very difficult to rehabilitate because the groundwater flow rate is very slow and low microbiological activities limit any self-purification processes which takes place in days or weeks in surface water systems can take decades in groundwater. Contamination can originate from point sources or non-point sources. Effect of contamination depends on nature and levels of the toxicants, sometimes causing serious health hazards even at low level, therefore most of the times remediation becomes a necessity for achieving sustainability.

In the subsurface water bodies, the transport of solute/solutes proceeds by mainly advection or hydrodynamic dispersion or molecular diffusion. Due to very slow velocity in subsurface water bodies, numerous attempts have been made to quantitatively describe the behavior of solute, adsorption, first-order decay and zero-order production by (van Genuchten and Alves, 1982; Lindstrom and Boersma, 1989). Ogata (1970) developed the theory of dispersion in granular media. Van Genuchten (1981) derived analytical solutions of one-dimensional solute transport problem with various boundary conditions. Matheron and deMarsily (1980) illustrated that in subsurface formations dispersivity may vary with position or time along uniform flow. Gelhar et al.(1992) demonstrate that dispersivity is depends on the time and/or scale which are not constant. Shan and Javandel (1997) developed analytical solutions for solute transport problems in a vertical section of a homogeneous aquifer with steady uniform flow. The source was assumed at the top (or the bottom) of the aquifer, and the initial concentration zero everywhere in the aquifer.

In the last few decades, an analytical model was presented for solute transport in rivers, including transient storage in hyporheic zones (Smedt, 2007), which consists of an ADE for transport in the main channel with a sink term describing diffusive solute transfer to the hyporheic zone. A classification of existing modeling approaches was proposed and theoretical models are reviewed with an emphasis on their mathematical formulations and their capacities to model the scale effect in longitudinal dispersion (Frippiat and Holeyman 2008). Chen (2008) presented an analytically of advection–dispersion equation with hyperbolic asymptotic distance-dependent dispersivity. Guerrero et al. (2009) also derived a formal exact solution of the linear advection-diffusion equation with constant coefficients for both transient and steady-state regimes using the classic version of Generalized Integral Transform Technique (GITT). This technique (GITT) is much used in recent works (Cassol et al. 2009; Moreira et al. 2009). Costa et al. (2006) derived advection-diffusion multilayer method for use in this context. Recently, Using Laplace Integral Transform Technique obtained an analytical solutions of dispersion problems (Jaiswal et al.2009, 2011; Kumar et al.2010) in which the two coefficients, one of which is dispersion parameter and the other is velocity of the flow, were considered as either temporally or spatially dependent or both. Gomes da Shilva et al. (2013) presented a new analytical approach for the solution of the advection-diffusion equation that describes a puff model for an instantaneous emission in non-homogeneous and non-stationary meteorological conditions. Swami et al. (2014) represented the combined effect of physical and chemical non-ideality in the stratified aquifer system for incorporation of distance dependent dispersivity in multi-process non-equilibrium model. Praveen et al. (2014) presented for simultaneous dispersion and adsorption of a solute within homogeneous and isotropic porous media in steady unidirectional flow fields. The solution is obtained with Laplace transform, Duhamel's theorem and moving coordinates to convert non-linear partial differential equation to ordinary differential equation. Karedla et al. (2014) presented a path-integral approach for finding solutions of the convection-diffusion equation with inhomogeneous fluid flow and compared the analytical approximation with numerical solutions which are obtained from a conventional finite-element time-difference method. Pereira et al. (2014) solved the one-dimensional advection-diffusion equation using an adaptive-step algorithm for the analysis of pollutant dispersion. Kumar et al. (2015) developed for conservative solute transport in a one-dimensional heterogeneous porous medium.

The objective of this study is to obtaining an analytical solution for two-dimensional advection-dispersion equation with variable coefficients. The domain is considered of semi-infinite extent and of heterogeneous nature along the both perpendicular directions of the plane. Due to heterogeneity of the medium, the velocity components of the flow transporting the solute particles through the medium along both the directions are considered spatially dependent. According to the theory

(Schiedegger, 1957), solute dispersion along both the directions is considered proportional to square of the respective velocity component. The input point source is considered of pulse type in both the cases. A flux type condition of homogeneous nature is considered at the infinite extent in both the directions. The medium is considered initially not solute free. Introduced new space variables and some other transformation to reduce our problem from two-dimensional ADE with variable coefficients to one-dimensional ADE with constant coefficient. Lastly the technique of Laplace Transform is used to obtain analytical solution. To have a better understood, the behavior of solute/solute transport shown graphically.

2. Methodology

Hydrodynamic dispersion of solute through a medium is described by a partial differential equation known as advection-diffusion equation (ADE). Mathematically, this partial differential equation is of parabolic type (Guenther and Lee, 1988; Logan, 1994) Eq. (3). Therefore, in the present problem we considered two-dimensional solute transport ADE through heterogeneous media.Integral Laplace transformation technique (ILTT) which is defined by Eq. (1) is used to get an analytical solution of said ADE. The Laplace transformation may be defined as;

If $f(x,t)$ is a any function defined in $a \le x \le b$ and $t > 0$, then its Laplace transform with respect to t is denoted by $L\{f(x,t)\} = F(x,p)$ and is defined by;

$$L\{f(x,t)\} = F(x,p) = \int_0^\infty e^{-pt} f(x,t)dt \quad , \quad p > 0 \tag{1}$$

where p is called the Laplace transform variable, which is a complex variable.

The inverse Laplace transform is denoted by $L^{-1}\{F(x,p)\} = f(x,t)$ and defined by the complex variable;

$$L^{-1}\{F(x,p)\} = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-pt} F(x,p)dp \quad c > 0 \tag{2}$$

3. Advection-Dispersion Equation

In two-dimensional space the linear advection-diffusion equation may be written as

$$\begin{aligned} \frac{\partial C}{\partial t} &= \frac{\partial}{\partial x}\left[D(x,t)\frac{\partial C}{\partial x} - u(x,t)C\right] - \gamma(x,t)C + \mu(x,t) \\ &+ \frac{\partial}{\partial y}\left[D(y,t)\frac{\partial C}{\partial y} - u(y,t)C\right] - \gamma(y,t)C + \mu(y,t) \end{aligned} \tag{3}$$

where C is the solute concentration at position x and time t in liquid phase, $D(x,t)$ and $D(y,t)$ be the solute dispersion parameter on the longitudinal and lateral (or transverse) directions and is called the dispersion coefficient if it is uniform and steady, $u(x,t)$ and $u(y,t)$ is the velocity of the medium transporting the solute particles along the longitudinal and lateral directions. $\gamma(x,t)$, $\mu(x,t)$, $\gamma(y,t)$ and $\mu(y,t)$ are first order decay and zero-order productions on both the directions respectively.

4. Analytical Solutions

The space dependence of steady velocity is considered due to heterogeneous\inhomogeneous of the medium. The inhomogeneity will cause variations in the velocity of the flow and dispersion. Such variation in the velocity is considered linearly increasing nature. It should be of small order so that the velocity at each position satisfies the laminar conditions of the flow domain, a necessary condition for this parameter occurring in the advection-diffusion equation. In the present section, longitudinal direction velocity increases from u_0 at $x = 0$ to $u_0(1+\lambda)$ at $x = L$, in a sufficiently large finite domain $0 \le x \le L$, due to inhomogeneous medium, where λ is a constant parameter. Hence longitudinal steady velocity may be linearly interpolated in this finite domain as

$$u(x,t) = u_{x0}(1 + \lambda x / L) = u_{x0}(1 + ax) \tag{4a}$$

where $a = \lambda / L$ is a parameter of dimension of $(length)^{-1}$. Similarly lateral direction, velocity may be linearly interpolated in a finite domain $0 \le y \le L$, due to inhomogeneous medium in terms of another parameter, b of dimension of $(length)^{-1}$ as:

$$u(y,t) = u_{y0}(1+by) \tag{4b}$$

It is also of increasing nature. Solute dispersion parameter is considered proportional to square of the velocity (Scheidegger, 1957) in both the longitudinal and lateral directions and hence it is considered that

$$D(x,t) = D_{x0}(1+ax)^2 \tag{5a}$$

and

$$D(y,t) = D_{y0}(1+by)^2 \tag{5b}$$

Let us write, $D(x,t)$, $D(y,t)$, $u(x,t)$, $u(y,t)$, $\gamma(x,t)$, $\mu(x,t)$, $\gamma(y,t)$ and $\mu(y,t)$ in Eq. (3) as follows

$$D(x,t) = D_{x0}(1+ax)^2;\ u(x,t) = u_{x0}(1+ax);\ \gamma(x,t) = \gamma_{x0};\ \mu(x,t) = \mu_{x0};$$

$$D(y,t) = D_{y0}(1+by)^2;\ u(y,t) = u_{y0}(1+by);\ \gamma(y,t) = \gamma_{y0} \text{ and } \mu(y,t) = \mu_{y0} \tag{5c}$$

respectively, where D_{x0} (L^2T^{-1}), D_{y0} (L^2T^{-1}), u_{x0} (LT^{-1}) and u_{y0} (LT^{-1}) are constants.

As ax and by are non-dimensional terms, hence the constants in Eqs. (4), D_{x0}, D_{y0} are longitudinal and lateral dispersion coefficients, respectively and u_{x0}, u_{y0} are longitudinal and lateral velocity components, respectively, in a homogeneous medium $(a=0,\ b=0)$. Both the constants a and b are chosen such that $0 < ax \le 1$ and $0 < by \le 1$.

Consequently the ADE in a space domain is given by Eq. (3) is,

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D_{x0}(1+ax)^2 \frac{\partial C}{\partial x} - u_{x0}(1+ax)C\right] - \gamma_{x0}C + \mu_{x0}$$

$$+\frac{\partial}{\partial y}\left[D_{y0}(1+by)^2 \frac{\partial C}{\partial y} - u_{y0}(1+by)C\right] - \gamma_{y0}C + \mu_{y0} \tag{6}$$

Now, introducing a new independent space variable, X (its dimension being the same as that of space variable x) by a transformation (Jaiswal et al. 2009, 2011, Kumar et al. 2010) as

$$X = -\int \frac{dx}{(1+ax)^2} \Leftrightarrow \frac{dX}{dx} = -\frac{1}{(1+ax)^2} \tag{7a}$$

$$\text{or } X = \frac{1}{a(1+ax)} \tag{7b}$$

$$\text{and } Y = -\int \frac{dy}{(1+by)^2} \Leftrightarrow \frac{dY}{dy} = -\frac{1}{(1+by)^2} \tag{8a}$$

$$\text{or } Y = \frac{1}{b(1+by)} \tag{8b}$$

Applying the transformation of Eqs. (7a,b) and (8a,b) on the partial differential equation, Eq. (6), becomes

$$\frac{\partial C}{\partial t} = a^2 D_{x0} X^2 \frac{\partial^2 C}{\partial X^2} + b^2 D_{y0} Y^2 \frac{\partial^2 C}{\partial Y^2} + a u_{x0} X \frac{\partial C}{\partial X} + b u_{y0} Y \frac{\partial C}{\partial Y} - \gamma_0 C + \mu_0 \tag{9}$$

where, $\gamma_0 = au_{x0} + bu_{y0} + \gamma_{x0} + \gamma_{y0}$, and $\mu_0 = \mu_{x0} + \mu_{y0}$.

Further, introducing another space variable by using a transformation

$$Z = -\log(aXbY) = \log\{(1+ax)(1+by)\} \tag{10}$$

Then the ADE Eq. (9), becomes

$$\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial Z^2} - U_0 \frac{\partial C}{\partial Z} - \gamma_0 C + \mu_0 \tag{11}$$

where $D_0=(a^2D_{x0}+b^2D_{y0})$, $U_0=(au_{x0}+bu_{y0}-D_0)$.

Now the analytical solutions are obtained for the both uniform and varying pulse type input point source, respectively. In the former case, the input concentration remains uniform up to a certain time period, beyond which the source of pollution is supposed to be eliminated hence the input concentration becomes zero. In the other case, varying pulse type input point source, the concentration increases in the presence of the source of pollution and once the source is eliminated, it starts decreasing.

4.1 Uniform input point source of pulse type condition

The initial concentration is assumed to be a decreasing function of position variables which tends to zero at the infinite extent along both the directions. The initial condition is written in mathematical form as

$$C(x,y,t)=\frac{C_i}{(1+ax)(1+by)}\,;\; t=0,\; x\ge 0,\; y\ge 0 \tag{12}$$

An input point source concentration of uniform nature and of pulse type is assumed to be introduced at the origin of the medium. It is assumed C_0 till $t=t_0$ and beyond that it is assumed zero. It means t_0 is the time when the point source is eliminated. The second boundary condition on both the direction at $x\to\infty$ and $y\to\infty$ is of second type (flux type) of homogeneous nature.

Thus the boundary conditions are as follows

$$C(x,y,t)=\begin{cases} C_0 & ;\ 0<t\le t_0; x=0, y=0 \\ 0 & ;\ t>t_0;\ x=0, y=0 \end{cases}, \tag{13}$$

$$\frac{\partial C}{\partial x}=0,\; \frac{\partial C}{\partial y}=0,\;\; x\to\infty, y\to\infty\; t\ge 0 \tag{14}$$

The initial and boundary conditions in Eqs. (12-14) may be written with the help of Eqs. (7a,b) and (8a,b) in terms of new independent variables (X,Y,t) domain as

$$C(X,Y,t)=C_i abXY\,;\; t=0,\; X\ge 0,\; Y\ge 0 \tag{15}$$

$$C(X,Y,t)=\begin{cases} C_0 & ;\ X=\frac{1}{a},\ Y=\frac{1}{b},\ 0<t\le t_0 \\ 0 & ;\ X=\frac{1}{a},\ Y=\frac{1}{b},\ t>t_0 \end{cases} \tag{16}$$

$$\frac{\partial C}{\partial X}=0,\; \frac{\partial C}{\partial Y}=0,\;\; X\to\infty,\; Y\to\infty\; t\ge 0 \tag{17}$$

With initial and boundary conditions Eqs. (15-17) may be written by Eq. (10) in (Z,t) domain as

$$C(Z,t)=C_i e^{-Z}\,;\; t=0,\; Z\ge 0 \tag{18}$$

$$C(Z,t)=\begin{cases} C_0 & ;\ 0<t\le t_0 \\ 0 & ;\ t>t_0 \end{cases};\; Z=0,\; t>0, \tag{19}$$

$$\frac{\partial C(Z,t)}{\partial Z}=0;\;\; Z\to\infty,\;\; t\ge 0. \tag{20}$$

Further, introduce a transformation by using Eq. (21)

$$C(Z,t)=K(Z,t)\exp\left\{\frac{U_0}{2D_0}Z-\left(\frac{U_0^2}{4D_0}+\gamma_0\right)t\right\}+\frac{\mu_0}{\gamma_0} \tag{21}$$

The advection-diffusion equation Eq. (11) reduces to diffusion equation with the help of Eq. (21) in (Z,t) domain, which is

$$\frac{\partial K(Z,t)}{\partial t}=D_0\frac{\partial^2 K(Z,t)}{\partial Z^2} \tag{22}$$

and the respective initial and boundary conditions Eqs. (18-20) may be written in terms of $K(Z,t)$, as follows

$$K(Z,t)=\left(C_i e^{-Z}-\frac{\mu_0}{\gamma_0}\right)\exp\left(-\frac{U_0}{2D_0}Z\right);\ \ t=0,\ \ Z\geq 0 \tag{23}$$

$$K(Z,t)=\begin{cases}\left(C_0-\dfrac{\mu_0}{\gamma_0}\right)\exp\{\alpha^2 t\}\ \ ;\ 0<t\leq t_0\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad ;\ Z=0,\ \ t>0\\ -\dfrac{\mu_0}{\gamma_0}\exp\{\alpha^2 t\}\qquad\quad ;\ t>t_0\end{cases} \tag{24}$$

$$\frac{\partial K(Z,t)}{\partial Z}+\frac{U_0}{2D_0}K(Z,t)=0;\ \ Z\to\infty,\ \ t\geq 0. \tag{25}$$

Where

$$\alpha^2=\left(\frac{U_0^2}{4D_0}+\gamma_0\right).$$

Now to eliminate time variable from right hand side of partial differential equation Eq.(22), *i.e.*, to reduce it into an ordinary differential equation of second order, let us multiply its both sides by $\exp(-pt)$ and integrate it between the limit $t=0$ and $t\to\infty$, *i.e.*, applying Laplace transformation of parameter p on equation (22), we get

$$\int_0^\infty \frac{\partial K(Z,t)}{\partial t}\exp(-pt)dt=D_0\int_0^\infty \frac{\partial^2 K(Z,t)}{\partial Z^2}\exp(-pt)dt$$

or

$$\left\{\left[K(Z,t)\exp(-pt)\right]_0^\infty+\int_0^\infty pK(Z,t)\exp(-pt)dt\right\}=D_0\frac{d^2\overline{K}(Z,t)}{dZ^2}$$

Using initial condition Eq. (23) on above, we have

$$p\overline{K}(Z,t)=D_0\frac{d^2\overline{K}(Z,t)}{dZ^2}+K(Z,0)$$

or

$$\left\{\left[K(Z,t)\exp(-pt)\right]_0^\infty+\int_0^\infty pK(Z,t)\exp(-pt)dt\right\}=D_0\frac{d^2\overline{K}(Z,t)}{dZ^2} \tag{26}$$

where

$$\overline{K}(Z,t)=\int_0^\infty \exp(-pt)K(Z,t)\,dt$$

and also the applying the Laplace transform on both the boundary conditions as follows

$$\overline{K}(Z,p)=\frac{C_0}{(p-\alpha^2)}\left[1-\exp\left\{-(p-\alpha^2)t_0\right\}\right]-\frac{\mu_0}{\gamma_0}\frac{1}{p-\alpha^2};\ \ Z=0, \tag{27}$$

where α first appears for the first time in Eq. (24),

and

$$\frac{d\overline{K}(Z,p)}{dZ}+\frac{U_0}{2D_0}\overline{K}(Z,p)=0;\ \ Z\to\infty. \tag{28}$$

where p is Laplace parameter is defined in Eq. (1).

Thus, the general solution of above mentioned ordinary differential equation Eq. (26) may be written as follows:

$$\bar{K}(Z,p)=c_1\exp\left\{-\sqrt{\frac{p}{D_0}}Z\right\}+c_2\exp\left\{\sqrt{\frac{p}{D_0}}Z\right\}+\frac{C_i}{\left(p-\delta^2\right)}\exp\{-\theta Z\}-\frac{\mu_0}{\gamma_0}\frac{\exp\left\{-\frac{U_0}{2D_0}Z\right\}}{\left(p-\beta^2\right)} \tag{29}$$

Where $\theta=\left(1+\frac{U_0}{2D_0}\right)$, $\beta^2=\frac{U_0^2}{4D_0}$, $\delta^2=D_0\theta^2$ and $\theta=D_0\left(1+\frac{U_0}{2D_0}\right)$.

Now, using both the boundary conditions to eliminate the arbitrary constants from the Eq. (29).
Using Eq. (28) in general solution Eq. (29), we may get
$c_2=0$ at $Z\to\infty$
And, using Eq. (27) in general solution Eq. (29), we may get

$$c_2=\frac{C_0}{(p-\alpha^2)}\left[1-\exp\left\{-(p-\alpha^2)t_0\right\}\right]-\frac{\mu_0}{\gamma_0}\left[\frac{1}{(p-\alpha^2)}-\frac{1}{(p-\beta^2)}\right]-\frac{C_i}{(p-\delta^2)} \text{ at } Z=0$$

Therefore, the general solution of ordinary differential equation Eq. (29) may be written as

$$\begin{aligned}\bar{K}(Z,p)=&\frac{C_0}{(p-\alpha^2)}\left[1-\exp\left\{-(p-\alpha^2)t_0\right\}\right]\exp\left\{-\sqrt{\frac{p}{D_0}}Z\right\}\\&-\frac{\mu_0}{\gamma_0}\left[\frac{1}{(p-\alpha^2)}-\frac{1}{(p-\beta^2)}\right]\exp\left\{-\sqrt{\frac{p}{D_0}}Z\right\}-\frac{C_i}{(p-\delta^2)}\exp\left\{-\sqrt{\frac{p}{D_0}}Z\right\}\\&+\frac{C_i}{\left(p-\delta^2\right)}\exp\{-\theta Z\}-\frac{\mu_0}{\gamma_0}\frac{\exp\left\{-\frac{U_0}{2D_0}Z\right\}}{\left(p-\beta^2\right)}.\end{aligned} \tag{30}$$

Now taking inverse Laplace transform of Eq. (30), the solution in $K(Z,t)$ may be obtained. For this purpose the table from van Genutchen and Alves (1982) has been used. The solution in $K(Z,t)$ may be written as

$$K(Z,t)=\left(C_0-\frac{\mu_0}{\gamma_0}\right)F(Z,t)+\frac{\mu_0}{\gamma_0}G(Z,t)+C_iH(Z,t);\quad 0<t\le t_0 \tag{31a}$$

$$K(Z,t)=\left(C_0-\frac{\mu_0}{\gamma_0}\right)F(Z,t)-C_0F(Z,t-t_0)+\frac{\mu_0}{\gamma_0}G(Z,t)+C_iH(Z,t);\quad t>t_0 \tag{31b}$$

where

$$F(Z,t)=\frac{1}{2}\exp\left\{\alpha^2t-\frac{\alpha Z}{\sqrt{D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{D_0t}}-\alpha\sqrt{t}\right\}+\exp\left\{\alpha^2t+\frac{\alpha Z}{\sqrt{D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{D_0t}}+\alpha\sqrt{t}\right\}$$

$$G(Z,t)=\frac{1}{2}\exp\left\{\beta^2t-\frac{\beta Z}{\sqrt{D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{D_0t}}-\beta\sqrt{t}\right\}+\exp\left\{\beta^2t+\frac{\beta Z}{\sqrt{D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{D_0t}}+\beta\sqrt{t}\right\}-\exp\left(-\frac{U_0}{2D_0}Z+\beta^2t\right)$$

and

$$\begin{aligned}H(Z,t)=\exp\left(-\theta Z+\delta^2t\right)-\frac{1}{2}&\left\{\exp\left\{\delta^2t-\frac{\delta Z}{\sqrt{D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{D_0t}}-\delta\sqrt{t}\right\}\right.\\&\left.+\exp\left\{\delta^2t+\frac{\delta Z}{\sqrt{D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{D_0t}}+\delta\sqrt{t}\right\}\right\}\end{aligned}$$

using back transformations Eqs. (21), (10), (8b) and (7b) as follows

$$C(x,y,t)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F_1(Z,t)-\frac{\mu_0}{\gamma_0}G_1(Z,t)+C_iH_1(Z,t);\ \ 0<t\le t_0 \tag{32a}$$

$$C(x,y,t)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F_1(Z,t)-C_0F_1(Z,t-t_0)-\frac{\mu_0}{\gamma_0}G_1(Z,t)+C_iH_1(Z,t)\ ;t>t_0 \tag{32b}$$

where

$$F_1(x,y,t)=\frac{1}{2}\exp\left\{\left(\frac{U_0-\sqrt{U_0^2+4\gamma_0D_0}}{2D_0}\right)Z\right\}erfc\left\{\frac{Z-\left(\sqrt{U_0^2+4\gamma_0D_0}\right)t}{2\sqrt{D_0t}}\right\}$$
$$+\frac{1}{2}\exp\left\{\left(\frac{U_0+\sqrt{U_0^2+4\gamma_0D_0}}{2D_0}\right)Z\right\}erfc\left\{\frac{Z+\left(\sqrt{U_0^2+4\gamma_0D_0}\right)t}{2\sqrt{D_0t}}\right\}$$

$$G_1(Z,t)=\exp\{-\gamma_0t\}\left[1-\frac{1}{2}erfc\left(\frac{Z-U_0t}{2\sqrt{D_0t}}\right)-\frac{1}{2}\exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{Z+U_0t}{2\sqrt{D_0t}}\right)\right]$$

and

$$H_1(Z,t)=\exp\{-Z+(D_0+U_0-\gamma_0)t\}$$
$$-\frac{1}{2}\exp\{-Z+(D_0+U_0-\gamma_0)t\}erfc\left\{\frac{Z-(2D_0+U_0)t}{2\sqrt{D_0t}}\right\}$$
$$-\frac{1}{2}\exp\left\{\left(1+\frac{U_0}{D_0}\right)Z+(D_0+U_0-\gamma_0)t\right\}erfc\left\{\frac{Z+(2D_0+U_0)t}{2\sqrt{D_0t}}\right\},$$

$Z=\log\{(1+ax)(1+by)\}$, $D_0=a^2D_{x0}+b^2D_{y0}$, $U_0=au_{x0}+bu_{y0}-D_0$, $\mu_0=\mu_{x0}+\mu_{y0}$ $\gamma_0=au_{x0}+bu_{y0}+\gamma_{x0}+\gamma_{y0}$

4.2 Varying input point source condition

The source of input concentration may increase with time due to variety of reasons. For example, the smoke is being released uniformly from a chimney into the atmosphere and is being driven along the wind. At a moment smoke stops coming out hence the concentration at the out let of the chimney becomes zero. A similar example may be given in a surface water body. But in case of infiltration of pollutants from the earth surface into an aquifer may increase due to increasing human activities. Once the source of pollution is removed, the infiltration hence the input concentration in the ground water domain decreases with time, instead of becoming zero at once. Accordingly, the input point source condition of pulse type defined by Eq. (9) is of uniform nature. It means in the presence of the source the input concentration remains uniform. As soon as the source is eliminated, it becomes zero. This more realistic scenario may be defined by a condition of mixed type of non-homogeneous nature which is

$$-D(x,t)\frac{\partial C(x,t)}{\partial x}+u(x,t)C(x,t)=\begin{cases}u_{x0}C_0 & ;\ 0<t\le t_0\\ 0 & ;\ t>t_0\end{cases};x=0 \tag{33a}$$

$$-D(y,t)\frac{\partial C(y,t)}{\partial y}+u(y,t)C(y,t)=\begin{cases}u_{y0}C_0 & ;\ 0<t\le t_0\\ 0 & ;\ t>t_0\end{cases};y=0 \tag{33b}$$

Using Eqs. (4a,b) and (5a,b) in the above equations, it may be written as

$$-D_{x0}(1+ax)^2\frac{\partial C(x,t)}{\partial x}+u_{x0}(1+ax)C(x,t)=\begin{cases}u_{x0}C_0 & ;\ 0<t\le t_0\\ 0 & ;\ t>t_0\end{cases};x=0 \tag{34a}$$

$$-D_{y0}(1+by)^2\frac{\partial C(y,t)}{\partial y}+u_{y0}(1+by)C(y,t)=\begin{cases}u_{y0}C_0 & ;\ 0<t\le t_0\\ 0 & ;\ t>t_0\end{cases};y=0 \tag{34b}$$

Again using the transformations Eqs. (7a,b), (8a,b) and Eq. (10) on above may be written as:

$$-D_0\frac{\partial C(Z,t)}{\partial Z}+U_1C(Z,t)=\begin{cases}U_1C_0 & ;\ 0<t\le t_0\\ 0 & ;\ t>t_0\end{cases};Z=0 \tag{35}$$

where $U_1=au_{x0}+bu_{y0}$.

Again, using the back transformation Eq. (21) in the above Eq. (35) may be written as

$$-D_0\frac{\partial K(Z,t)}{\partial Z}+\frac{U_2}{2}K(Z,t)=\begin{cases}U_1\left(C_0-\frac{\mu_0}{\gamma_0}\right)\exp\left\{\alpha^2t\right\} & ;\ 0<t\le t_0\\ -U_1\frac{\mu_0}{\gamma_0}\exp\left\{\alpha^2t\right\} & ;\ t>t_0\end{cases};Z=0 \tag{36}$$

where $U_2=2U_1+U_0$.

Applying the Laplace transformation on above, we may get

$$-D_0\frac{d\bar{K}(Z,p)}{dZ}+\frac{U_2}{2}\bar{K}(Z,p)=\frac{U_1C_0}{(p-\alpha^2)}\left[1-\exp\left\{-\left(p-\alpha^2\right)t_0\right\}\right]-\frac{U_1\mu_0}{\gamma_0}\frac{1}{(p-\alpha^2)};\ Z=0 \tag{37}$$

Now using the condition Eq. (37) in place of Eq. (27), for eliminating the arbitrary constants c_1 and c_2 in Eq. (29), the particular solution of Eq. (26) satisfying the conditions in Eqs. (37) and (28), may be written as

$$\begin{aligned}\bar{K}(Z,p)=&\frac{U_1C_0}{\sqrt{D_0}}\frac{\left[1-\exp\left\{-\left(p-\alpha^2\right)t_0\right\}\right]}{\left(p-\alpha^2\right)\left(\sqrt{p}+\phi\right)}\exp\left\{-Z\sqrt{p/D_0}\right\}\\&-\frac{U_1\mu_0}{\sqrt{D_0}\gamma_0}\frac{\exp\left\{-Z\sqrt{p/D_0}\right\}}{\left(p-\alpha^2\right)\left(\sqrt{p}+\phi\right)}-\theta C_i\sqrt{D_0}\frac{\exp\left\{-Z\sqrt{p/D_0}\right\}}{\left(p-\delta^2\right)\left(\sqrt{p}+\phi\right)}\\&-\frac{U_2C_i}{2\sqrt{D_0}}\frac{\exp\left\{-Z\sqrt{p/D_0}\right\}}{\left(p-\delta^2\right)\left(\sqrt{p}+\phi\right)}+\frac{\mu_0\left(U_2+U_0\right)}{2\gamma_0\sqrt{D_0}}\frac{\exp\left\{-Z\sqrt{p/D_0}\right\}}{\left(p-\beta^2\right)\left(\sqrt{p}+\phi\right)}\\&+\frac{C_i\exp\left(-\theta Z\right)}{\left(p-\delta^2\right)}-\frac{\mu_0}{\gamma_0}\frac{\exp\left\{-\frac{U_0}{2D_0}Z\right\}}{\left(p-\beta^2\right)}\end{aligned} \tag{38}$$

Where $\phi^2=\frac{U_2^2}{4D_0}$, $\beta^2=\frac{U_0^2}{4D_0}$, $\theta=\left(1+\frac{U_0}{2D_0}\right)$ and $\delta^2=D_0\left(1+\frac{U_0}{2D_0}\right)^2=D_0\theta^2$.

Now, applying inverse Laplace transform on Eq. (38) and using back transformations Eqs. (21), (10), (8) and (7), to get the desired solution of advection-diffusion equation for input point source for varying nature may be written in terms of $C(x,y,t)$ as follows

$$\begin{aligned}C(Z,t)=&\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)L(Z,t)-C_i\left(U_1+D_0\right)M(Z,t)+\frac{\mu_0}{\gamma_0}N(Z,t)\\&+C_i\exp\left\{-Z+\left(D_0+U_0-\gamma_0\right)t\right\};\ 0<t\le t_0\end{aligned} \tag{39a}$$

$$C(Z,t)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)L(Z,t)-C_0L(Z,t-t_0)-C_i\left(U_1+D_0\right)M(Z,t)+\frac{\mu_0}{\gamma_0}N(Z,t)$$
$$+C_i\exp\left\{-Z+\left(D_0+U_0-\gamma_0\right)t\right\};\ t>t_0 \tag{39b}$$

where

$$L(Z,t)=\frac{U_1}{U_2+\sqrt{U_0^2+4\gamma_0D_0}}\exp\left\{\left(\frac{U_0-\sqrt{U_0^2+4\gamma_0D_0}}{2D_0}\right)Z\right\}erfc\left\{\left(\frac{Z-\sqrt{U_0^2+4\gamma_0D_0t}}{2\sqrt{D_0t}}\right)\right\}$$
$$+\frac{U_1}{U_2-\sqrt{U_0^{\,2}+4\gamma_0D_0}}\exp\left\{\left(\frac{U_0+\sqrt{U_0^{\,2}+4\gamma_0D_0}}{2D_0}\right)Z\right\}erfc\left\{\left(\frac{Z+\sqrt{U_0^{\,2}+4\gamma_0D_0\,t}}{2\sqrt{D_0\,t}}\right)\right\}$$
$$+\frac{2U_2U_1}{U_0^2+4\gamma_0D_0-U_2^2}exp\left\{\left(\frac{U_0+U_2}{2D_0}\right)Z+\left(\frac{U_2^2-U_0^2-4\gamma_0D_0}{4D_0}\right)t\right\}erfc\left(\frac{Z+U_2t}{2\sqrt{D_0t}}\right),$$

$$M(Z,\mathrm{t})=\frac{1}{2D_0+U_0+U_2}\exp\left\{-Z+\left(\mathrm{D}_0+\mathrm{U}_0-\gamma_0\right)\mathrm{t}\right\}.erfc\left(\frac{Z-\left(2+U_0/D_0\right)t}{2\sqrt{D_0t}}\right)$$
$$-\frac{1}{2D_0+U_0-U_2}exp\left\{\left(1+\frac{U_0}{D_0}\right)Z+\left(D_0+U_0-\gamma_0\right)t\right\}.erfc\left(\frac{Z+\left(2+U_0/D_0\right)t}{2\sqrt{D_0t}}\right)$$
$$+\frac{2U_2}{(2D_0+U_0)^2-U_2^2}\exp\left\{\left(\frac{U_0+U_2}{2D_0}\right)Z+\left(\frac{U_2^2-\left(U_0^2+4\gamma_0D_0\right)}{2D_0}\right)t\right\}.erfc\left(\frac{Z+U_2t}{2\sqrt{D_0t}}\right),$$

$$N(Z,t)=exp\left(-\gamma_0t\right)\left[-1+\frac{1}{2}erfc\left(\frac{Z-U_0t}{2\sqrt{D_0t}}\right)-\frac{U_2+U_0}{2(U_2-U_0)}exp\left(\frac{U_0}{D_0}Z\right)erfc\left(\frac{Z+U_0t}{2\sqrt{D_0t}}\right)\right]$$
$$-\frac{U_2}{(U_0-U_2)}\exp\left(\frac{U_0+U_2}{2D_0}Z+\left(\frac{U_2^2-U_0^2-4\gamma_0D_0}{4D_0}\right)t\right)erfc\left(\frac{Z+U_2t}{2\sqrt{D_0t}}\right),$$

$U_2=2U_1-U_0,\ U_1=au_{x_0}+bu_{y_0}$,

$Z=\log\left\{(1+ax)(1+by)\right\}$, $D_0=a^2D_{x0}+b^2D_{y0}$, $U_0=au_{x0}+bu_{y0}-D_0$, $\mu_0=\mu_{x0}+\mu_{y0}$,

$\gamma_0=au_{x0}+bu_{y0}+\gamma_{x0}+\gamma_{y0}$.

5. Illustrations and Discussions

In order to illustrate the clear in the proposed problem, analytical result is demonstrated with the help of graphs. Analytical solution for uniform and varying pulse type input point source is given by Eqs. (32a, b) and Eqs. (39a ,b), respectively. To analyzed the concentration distribution behaviors with help a numerical example. Some hypothetical parameters are used to demonstrate the concentration behavior. An analytical solution obtained as in Eqs. (32a, b), and its particular solutions discussed for a chosen set of data taken from the experimental and theoretical literatures. The concentration values $\left(C/C_0\right)$ are evaluated assuming reference concentration as $C_0=1.0$, in a finite domain along longitudinal and transverse directions $0\le x(\text{km})\le 1$, $0\le y(\text{km})\le 1$, respectively i.e., lengths of the finite domain have been taken $L=1.0$ (km) in longitudinal and lateral directions. The inhomogeneity parameters a (km^{-1}) and b (km^{-1}) are also considered in both the directions. It is represented by a set of values $a=b=1.0$ (km^{-1}). As in most of the literatures the lateral (transverse) component of velocity is considered negligible compared to longitudinal velocity in a medium of substantial depth hence it is chosen almost one-tenth of the longitudinal component to see its extreme effect on the solute transport. The values for initial velocity and dispersion coefficient are taken as $u_{x0} = 1.15$ (km/year), $u_{y0} = 0.15$ (km/year), $D_{x0} = 1.25$ (km^2/year), $D_{y0} = 0.25$ (km^2/year), respectively. In addition to these, first order decay (γ) and zero order production (μ) are also considered in both the directions as $\gamma_{x0} = \gamma_{y0} = 0.015$, $\mu_{x0} = \mu_{y0} = 0.05$. The source of pollution is supposed to be eliminated at a time given by $t_0 = 1.0$ (year).

In the presence of the source of pollution ($t < t_0$), i.e., in the time taken into an account at $t(\text{year}) = 0.35$, 0.85. The input concentration, (C/C_0) at the origin, $x = 0$, $y = 0$ is equal to one at each time. It attenuates with position and time. Fig. 1a illustrates the solute transport for the uniform input point source along the longitudinal and lateral directions of the medium, described by the solution in Eq. (32a), Similarly Fig. 1b illustrates the solute transport described by the solution in Eq. (32b), once the source of the pollution is eliminated, i.e., the input concentration remains zero in the time domain ($t > t_0$) at $t(\text{year}) = 1.35$, 1.85. Figs 1a,b shows the trend with which the polluted domain gets concentration free with position and time. It may be observed that, in the presence of the source of pollutant the concentration are increased with increasing time at a particular position and after the source of pollutant is eliminated, the concentration are decreased with increasing time at a particular position.

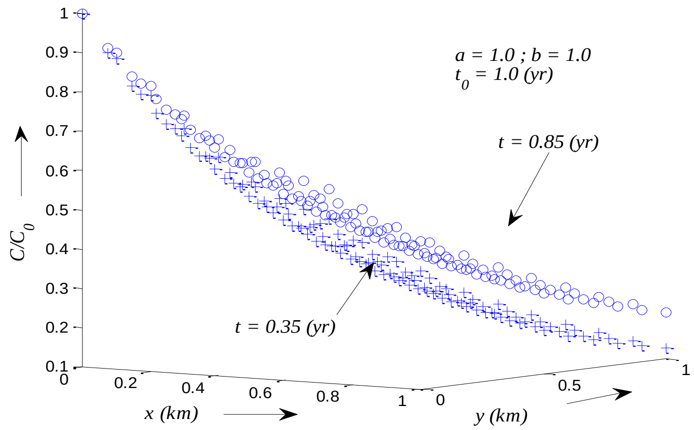

Figure 1a. Solute concentration distribution behavior for uniform pulse type input point source described by solution (32a) in the presence of the source of pollutant ($t<t_0$)

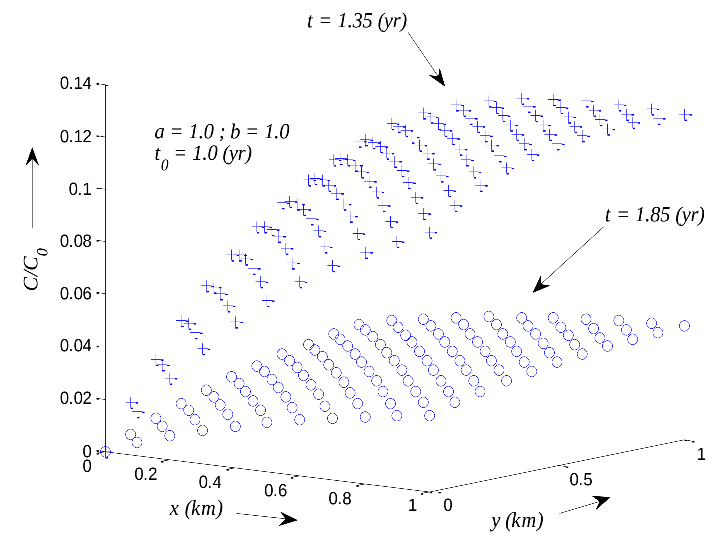

Figure 1b. Solute concentration distribution behavior for uniform pulse type input point source described by solution (32b) in the absence of the source of pollutant ($t>t_0$).

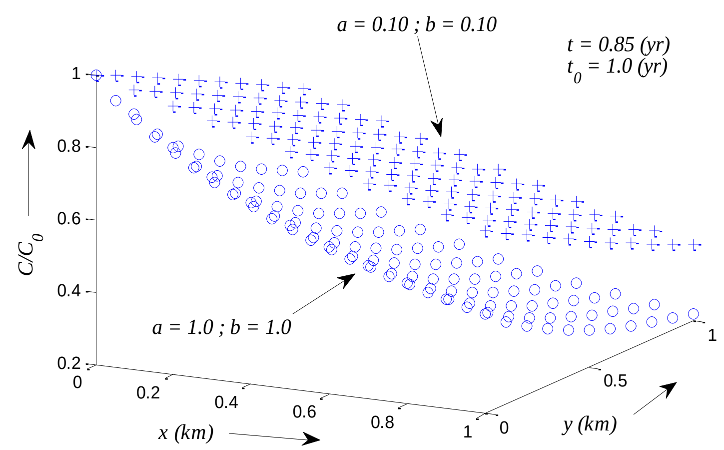

Figure 2a. Comparison of solute concentration distribution behavior for uniform pulse type input point source described by solution (32a) in the presence of the source of pollutant ($t<t_0$) at different inhomogeniety parameters

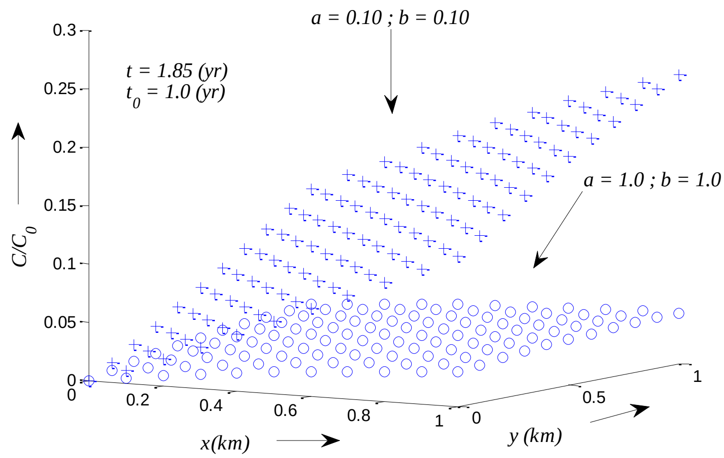

Figure 2b. Comparison of solute concentration distribution behavior for uniform pulse type input point source described by solution (32b) in the absence of the source of pollutant ($t>t_0$) at different inhomogeniety parameters.

Figs 2a and 2b, shows the concentration profile which are evaluated from the solutions (32a) and (32b) are at $t = 0.85(\text{yrs})$ and $t = 1.85(\text{yrs})$, respectively, treating $t_0 = 1.0(\text{yrs})$, for the inhomogeneity parameters $a = b = 0.1\,(\text{km})^{-1}$ and compared with different inhomogeneity values $a = b = 1.0\,(\text{km})^{-1}$, at the same respective times. Fig 2a, shows in the presence of source of pollutant, the concentration profile in a lower inhomogeneity is higher than that in the medium of higher inhomogeneity. Fig. 2b reveals that once the point source is eliminated the rehabilitation process becomes slower in a medium of lower inhomogeneity than that in a medium of higher inhomogeneity.

Further the solutions (39a) and (39b) which describe the solute dispersion of a pulse type varying point source are illustrated in Figs 3a and 3b, respectively. It may be observed that input concentration (C/C_0) at $x = 0$, $y = 0$ increases with time in the presence of the source and decreases with time in the absence of the source. The time of elimination of the source is considered at $t_0 = 1.0$ (yrs). The concentration distribution patterns of varying pulse type input point source in two media of different inhomogeneity, are also compared with each other in Figs 4a and 4b. For this the concentration value from solutions (39a) and (39b) are evaluated at $t = 0.85(\text{yrs})$ and $t = 1.85(\text{yrs})$, respectively, treating $t_0 = 1.0(\text{yrs})$, for the inhomogeneity parameters $a = b = 0.1\,(\text{km})^{-1}$. These values are compared with those obtained for the values $a = b = 1.0\,(\text{km})^{-1}$, at the same respective times. The Figs 4a and 4b shows that in the presence and absence of varying pulse type point source. The concentration transport pattern in a medium is same tendency as in for uniform pulse type point source.

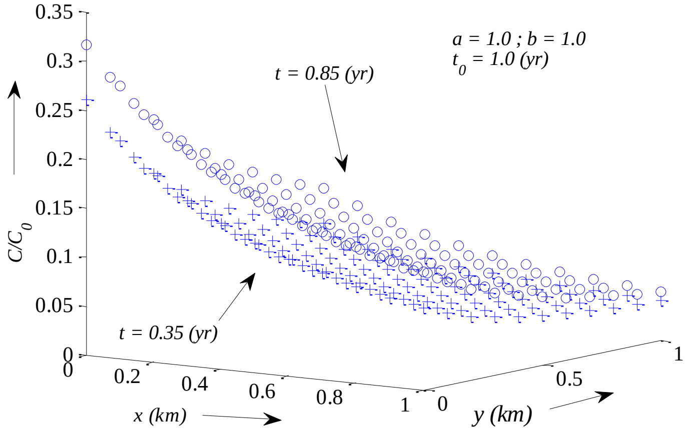

Figure 3a. Solute concentration distribution behavior for varying pulse type input point source described by solution (39a) in the presence of the source of pollutant ($t<t_0$)

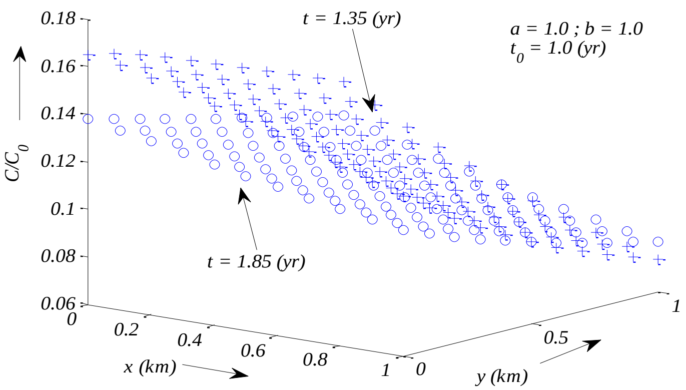

Figure 3b. Solute concentration distribution behavior for varying pulse type input point source described by solution (39b) in the absence of the source of pollutant ($t>t_0$).

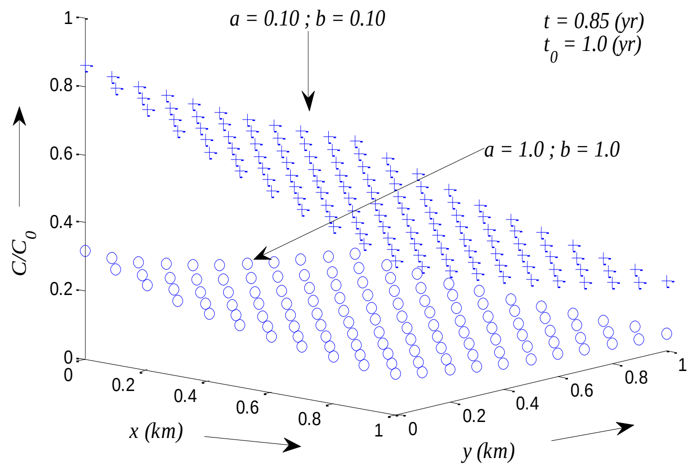

Figure 4a. Comparison of solute concentration distribution behavior for varying pulse type input point source described by solution (39a) in the presence of the source of pollutant ($t<t_0$) at different inhomogeniety parameters.

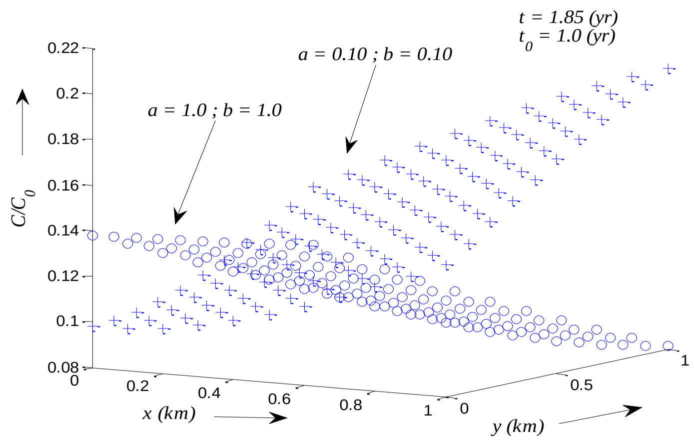

Figure 4b. Comparison of solute concentration distribution behavior for uniform pulse type input point source described by solution (39b) in the absence of the source of pollutant ($t>t_0$) at different inhomogeniety parameters.

6. Conclusions

In the present problem, two-dimensional advection-diffusion equation in a horizontal plane is considered. The steady velocity component in each direction is linearly interpolated in respective position variable, increasing up to a finite length. This scale-dependence is due to inhomogeneous nature of the medium. According to the theory of Scheidegger (1957), $D=\rho u^2$ (dispersion being proportional to the square of velocity) has been implemented "Solute dispersion component in each direction is considered proportional to the respective velocity component". The medium is not solute free at the initial stage. Such problems are describe the solute transport of a pulse type input point sources of uniform and varying nature both, along the spatially dependent flow domains. The variable coefficients of the advection-diffusion equation are reduced into constant coefficients with the help of new independent variables introduced through different transformations. The analytical solutions are obtained by using the technique of Laplace transformation. The effect of inhomogeneity on the solute transport is also shown. It may be observed that though the values of transverse component of dispersion coefficient and velocity are taken almost one-tenth of respective longitudinal component value, yet the solute transport along lateral direction is also significant. The figures are drawn using MATLAB to illustrate the analytical solutions.

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