A New Intelligent Method for Eliminating Unknown (Nuisance) Parameters from Underlying Models as an Alternative to the Bayesian Approach

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Abstract

The big question in statistics is: How can we eliminate the unknown (nuisance) parameter from an underlying model? Eliminating unknown (nuisance) parameters from an underlying model is universally recognized as a major problem of statistics and has been formally studied in virtually all approaches to inference. A surprisingly large number of elimination methods have been proposed in the literature on this topic. The classical method of elimination of unknown (nuisance) parameters from the model, which is used repeatedly in the large sample theory of statistics, is to replace the unknown (nuisance) parameter by an estimated value. However, this method is not efficient when dealing with small data samples. The Bayesian approach is dependent of the choice of priors. In this paper, a new method is proposed to eliminate the unknown (nuisance) parameter from the underlying model. This method isolates and eliminates unknown (nuisance) parameters from the underlying model as efficiently as possible. Unlike the Bayesian approach, the proposed method is independent of the choice of priors and represents a novelty in the theory of statistical decisions. It allows one to eliminate unknown parameters from the problem and to find the efficient statistical decision rules, which often have smaller risk than any of the well-known decision rules. To illustrate the proposed method, some practical applications are given.

Keywords

Underlying model, Parametric uncertainty, Elimination of unknown (nuisance) parameters, Efficient statistical decisions

1. Introduction

The problem of determining an optimal decision rule in the absence of complete information about the underlying distribution, i.e., when we specify only the functional form of the distribution and leave some or all of its parameters unspecified, is seen to be a standard problem of statistical estimation. Unfortunately, the classical theory of statistical estimation has little to offer in general type of situation of loss functions. Eliminating unknown (nuisance) parameters from an underlying model is widely recognized as one of the main problems of statistics. A surprisingly large number of elimination methods have been proposed by various authors on this topic. The big question in statistics is: How can we exclude a nuisance (unknown) parameter from an underlying model? An incredible amount of effort and ingenuity has been ex-

pendent in finding reasonable answers to this question [1, 2]. In this paper, an innovative statistical pivot-based approach to elimination of unknown (nuisance) parameters from the underlying models is proposed for constructing efficient statistical decisions under parametric uncertainty. The main aim of the present paper is to show how the innovative statistical pivot-based approach may be employed in the particular case of constructing the exact statistical prediction limits and shortest-length or equal tails confidence intervals for future random quantities under parametric uncertainty. The approach used here is a special case of more general considerations applicable whenever the statistical problem is invariant under a group of transformations, which acts transitively on the parameter space [3-18].

2. Statistical Prediction of Future Random Quantities

Statistical prediction of future random quantities (future outcomes, order statistics, etc.) based on the past and current data is the most prevalent form of statistical inference. Predictive inferences for future random quantities are widely used in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications, and many other industries. Predictive inferences (predictive distributions, prediction limits, confidence (prediction) intervals) for future random quantities on the basis of the past and present knowledge represent a fundamental problem of statistics, arising in many contexts and producing varied solutions. Statistical prediction is the process by which values for unknown observables (potential observations yet to be made or past ones which are no longer available) are inferred based on current observations and other information at hand.

3. Types of Statistical Prediction Problems

There are the following types of prediction problems:

A. New-sample prediction problem. In this case, the data from a past sample of size n are used to make prediction on one or more future units in a second sample of size m from the same process or population. For example, based on previous (possibly censored) life test data, one could be interested in predicting the following: (1) time to failure of a new item ($m = 1$); (2) time until the k th failure in a future sample of m units, $m \geq k$; (3) number of failures by time τ^* in a future sample of m units. Formally, we call the problems in this category as two-sample problems.

B. Within-sample prediction problem. In this case, the problem is to predict future events in a sample or process based on the early-failure data from that sample or process. For example, if n units are followed until censoring time τ_c and there are r observed ordered failure times, $X_1 \leq \dots \leq X_r$, one could be interested in predicting the following: 1) time of next failure; 2) time until l additional failures, $l \leq n - r$; 3) number of additional failures in a future interval (τ_c, τ^*) . Formally we call the problems in this category as one-sample problems.

C. New-within-sample prediction problem. In this case, the problem is to predict future events in a sample or process based on both the early-failure data from that sample or process and the data from a previous sample (i.e., when for predicting a future failure time of a unit in a new sample there are available both the early-failure data from that sample and the data from a previous sample). Formally we call the problems in this category as multi-sample problems.

In this paper, the within-sample prediction problems are considered.

4. Types of Statistical Prediction Limits

Usually, two types of statistical prediction limits are defined: i) statistical γ -content prediction limits with expected $(1-\alpha)$ -confidence on future outcomes, ii) statistical prediction limits with expected $(1-\alpha)$ -confidence on future outcomes (or simply $(1-\alpha)$ -prediction limits). The statistical γ -content prediction limit with expected $(1-\alpha)$ -confidence seems to be more useful than the statistical prediction limit with expected $(1-\alpha)$ -confidence but is relatively difficult to construct.

5. Pivot-Based Method for Eliminating Unknown (Nuisance) Parameters from Underlying Models

The pivot-based method for eliminating unknown (nuisance) parameters from the underlying model includes the following 3 stages:

Stage 1 (*Invariant statistical embedding of a sample statistic in the underlying model to construct a pivotal quantity and isolate unknown (nuisance) parameter from the problem*). At this stage, an invariant embedding of a sample statistic in the underlying model (say, probability distribution, decision criterion (performance index), etc.) is carried out to construct a *pivotal quantity* (or simply a *pivot*) in order to isolate the unknown (nuisance) parameter from the problem, since the pivot's probability distribution does not depend on the unknown parameter.

Stage 2 (Averaging the underlying model over the probability distribution of the pivotal quantity to eliminate unknown (nuisance) parameter from the problem). At this stage, the underlying model (say, probability distribution, decision criterion (performance index), etc.) is averaged over the probability distribution of the pivotal quantity in order to eliminate the unknown (nuisance) parameter from the problem.

Stage 3 (Efficient decision-making process). At this stage, when the unknown (nuisance) parameter has been eliminated from the underlying model (say, probability distribution, decision criterion (performance index), etc.), it can be found an efficient statistical decision rule.

6. Exponential Distribution

The one-parameter exponential distribution is one of the most significant and widely used distribution in statistical practice. It possesses several important statistical properties, and yet exhibits great mathematical tractability. This distribution is one of the most commonly used models in life-testing and reliability studies.

Let $\mathbf{Y}=(Y_1 \leq \dots \leq Y_k)$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution with the probability density function

$$f_{\beta}(y) = \beta^{-1} \exp(-y / \beta), \quad \beta > 0, y > 0, \quad (1)$$

and the cumulative distribution function

$$F_{\beta}(y) = 1 - \exp(-y / \beta), \quad (2)$$

where β is the scale parameter. It is assumed that the parameter β is unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and Y_k is a random variable. The MLE's $\hat{\beta}$ of the parameter β is given by

$$\hat{\beta} = \frac{S_k}{k} = \frac{\sum_{i=1}^k Y_i + (m-k)Y_k}{k}, \quad (3)$$

It is known that S_k is the complete sufficient statistic for β . Then

$$V_k = S_k / \beta \quad (4)$$

is the pivotal quantity, the probability density function of which is given by

$$f(v_k | k) = \frac{1}{\Gamma(k)} v_k^{k-1} \exp(-v_k), \quad v_k \geq 0. \quad (5)$$

The probability density function of the order statistic Y_k is given by

$$g_{\beta}(y_k) = \frac{m!}{(k-1)!(m-k)!} [F_{\beta}(y_k)]^{k-1} [1 - F_{\beta}(y_k)]^{m-k} f_{\beta}(y_k) = \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} \times (-1)^j \exp\left(-\frac{y_k}{\beta}(m-k+1+j)\right) \frac{1}{\beta}, \quad y_k \in (0, \infty). \quad (6)$$

The cumulative distribution function of the k th order statistic Y_k is given by

$$G_{\beta}(y_k) = \Pr(Y_k \leq y_k) = \int_0^{y_k} g_{\beta}(y) dy = 1 - \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{\exp\left[-\frac{y_k}{\beta}(m-k+j+1)\right]}{m-k+j+1}. \quad (7)$$

It follows from (7) that the cumulative distribution function of the pivotal quantity

$$V_k^{\bullet} = \frac{Y_k}{\beta} \quad (8)$$

is given by

$$G(v_k^*) = 1 - \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{\exp[-v_k^*(m-k+j+1)]}{m-k+j+1}. \quad (9)$$

The probability density function of V_k^* is given by

$$g(v_k^*) = \frac{dG(v_k^*)}{dv_k^*} = \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} \times (-1)^j \exp(-v_k^*(m-k+j+1)), \quad v_k^* \in (0, \infty). \quad (10)$$

Theorem 1 (Conditional probability distribution of the l th order statistic). Let $Y_1 \leq \dots \leq Y_k$ be the first k ordered observations (order statistics) in a sample of size m from a continuous distribution (say, exponential) with some probability density function $f_\beta(y)$ and distribution function $F_\beta(y)$, where β is a parameter. Then the conditional probability distribution function of Y_l ($1 \leq l < k \leq m$) given $Y_k = y_k$ is determined as

$$\begin{aligned} \Pr(Y_l \leq y_l | Y_1 = y_1, \dots, Y_k = y_k) &= \Pr(Y_l \leq y_l | Y_k = y_k) = 1 - \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \left[\frac{\bar{F}_\beta(y_l)}{\bar{F}_\beta(y_k)} \right]^{m-l+1+j} \\ &= 1 - \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \times \exp\left(-\frac{y_l - y_k}{\beta}(m-l+1+j)\right) = G_\beta(y_l | y_k). \end{aligned} \quad (11)$$

where

$$\bar{F}_\beta(y) = 1 - F_\beta(y). \quad (12)$$

Proof. The proof is omitted here and will appear elsewhere.

It follows from (11) that

$$\begin{aligned} \Pr(Y_l > y_l | Y_k = y_k) &= 1 - \Pr(Y_l \leq y_l | Y_k = y_k) = \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \\ &\times \exp\left(-\frac{y_l - y_k}{\beta}(m-l+1+j)\right) = \bar{G}_\beta(y_l | y_k). \end{aligned} \quad (13)$$

It follows from (11) that the cumulative distribution function of a pivotal quantity

$$V_{lk} = \frac{Y_l - Y_k}{\beta} \quad (14)$$

is given by

$$G(v_{lk}) = 1 - \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \exp(-v_{lk}(m-l+1+j)). \quad (15)$$

The probability density function of the pivotal quantity V_{lk} is given by

$$g(v_{lk}) = \frac{dG(v_{lk})}{dv_{lk}} = \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \exp(-v_{lk}(m-l+1+j)), \quad v_{lk} \geq 0. \quad (16)$$

7. Statistical Pivot-Based Estimation of Probability Distributions

Suppose Y is a future observation from the same distribution (1), independent of $\mathbf{Y} = (Y_1 \leq \dots \leq Y_k)$. Then the pivot-based estimate of (2) can be determined as follows:

Step 1. Invariant embedding of S_k in (2) to isolate unknown parameter β from the problem through V_k ,

$$F_\beta(y) = 1 - \exp(-y/\beta) = 1 - \exp\left(-\frac{y}{s_k} \frac{s_k}{\beta}\right) = 1 - \exp\left(-\frac{y}{s_k} v_k\right) = F_\beta\left(\frac{y}{s_k} v_k\right), \quad (17)$$

where

$$W = \frac{Y}{S_k} \quad (18)$$

is an ancillary statistic.

Step 2. Averaging (17) over the probability distribution of the pivotal quantity V_k to eliminate unknown parameter β from the problem. It follows from (5) and (17) that the pivot-based estimate of the cumulative distribution function (2) (through the pivot-based approach) is given by

$$F_{s_k}(y) = \int_0^\infty F_\beta\left(\frac{y}{s_k}v_k\right)f(v_k|k)dv_k = \int_0^\infty \left[1 - \exp\left(-\frac{y}{s_k}v_k\right)\right] \frac{1}{\Gamma(k)}v_k^{k-1}\exp(-v_k)dv_k = 1 - \frac{1}{\left(1 + \frac{y}{s_k}\right)^k}. \quad (19)$$

The pivot-based estimate of the probability density function (1) is given by

$$f_{s_k}(y) = \frac{dF_{s_k}(y)}{dy} = \frac{k}{s_k} \left(1 + \frac{y}{s_k}\right)^{-k-1}, \quad y \geq 0, \quad (20)$$

and

$$\bar{F}_{s_k}(y) = 1 - F_{s_k}(y) = \left(1 + \frac{y}{s_k}\right)^{-k}. \quad (21)$$

It follows from (19) that the cumulative distribution function of the ancillary statistic (18) is given by

$$F(w) = 1 - \frac{1}{(1+w)^k}. \quad (22)$$

The probability density function of the ancillary statistic (18) is given by

$$f(w) = \frac{dF(w)}{dw} = \frac{k}{(1+w)^{k+1}}, \quad w \geq 0. \quad (23)$$

It follows from (5) and (7) that the pivot-based estimate of the cumulative distribution function of the k th order statistic Y_k is given by

$$G_{s_k}(y_k) = \int_0^\infty G_\beta\left(\frac{y_k}{s_k}v_k\right)f(v_k|k)dv_k = 1 - \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{\left[1 + (m-k+j+1)\frac{y_k}{s_k}\right]^{-k}}{m-k+j+1}, \quad (24)$$

where

$$W_k = \frac{Y_k}{S_k} \quad (25)$$

is an ancillary statistic. It follows from (24) that

$$\bar{G}_{s_k}(y_k) = 1 - G_{s_k}(y_k) = \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{\left[1 + (m-k+j+1)\frac{y_k}{s_k}\right]^{-k}}{m-k+j+1}. \quad (26)$$

It follows from (24) that the cumulative distribution function of the ancillary statistic W_k is given by

$$G(w_k) = 1 - \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{\left[1 + (m-k+j+1)w_k\right]^{-k}}{m-k+j+1}, \quad (27)$$

The probability density function of the ancillary statistic W_k is given by

$$g(w_k) = \frac{dG(w_k)}{dw_k} = \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j k [1 + (m-k+j+1)w_k]^{-k-1}, \quad w_k \geq 0. \quad (28)$$

It follows from (5) and (11) that the pivot-based estimate of the conditional probability distribution function of the l th order statistic Y_l ($1 \leq k < l \leq m$) given $Y_k=y_k$ is determined (by using the pivot-based method for elimination of the unknown parameter β from (11) via $S_k = s_k$) as

$$G_{s_k}(y_l | y_k) = 1 - \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \frac{\left[1 + (m-l+1+j) \frac{y_l - y_k}{s_k}\right]^{-k}}{m-l+1+j}, \quad (29)$$

where

$$W_{lk} = \frac{Y_l - Y_k}{S_k} \quad (30)$$

is an ancillary statistic. It follows from (29) that

$$\bar{G}_{s_k}(y_l | y_k) = \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \frac{\left[1 + (m-l+1+j) \frac{y_l - y_k}{s_k}\right]^{-k}}{m-l+1+j}, \quad (31)$$

It follows from (29) that the cumulative distribution function of the ancillary statistic W_{lk} is given by

$$G(w_{lk}) = 1 - \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \frac{[1 + (m-l+1+j)w_{lk}]^{-k}}{m-l+1+j}, \quad (32)$$

The probability density function of the ancillary statistic W_{lk} is given by

$$g(w_{lk}) = \frac{dG(w_{lk})}{dw_{lk}} = \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j k [1 + (m-l+1+j)w_{lk}]^{-k-1}, \quad w_{lk} \geq 0. \quad (33)$$

It follows from (11) that the conditional probability distribution function of the l th order statistic Y_l ($1 \leq k < l \leq m$) given $Y_k=y_k$ can be presented as

$$G_{\beta}(y_l | y_k) = 1 - \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \exp\left(-\frac{y_l - y_k}{y_k} (m-l+1+j) \frac{y_k}{\beta}\right), \quad (34)$$

where

$$W_{lk}^{\bullet} = \frac{Y_l - Y_k}{Y_k} \quad (35)$$

is an ancillary statistic. It follows from (10) and (34) that the pivot-based estimate of the conditional cumulative distribution function of the l th order statistic Y_l ($1 \leq k < l \leq m$) given $Y_k=y_k$ is determined (by using the pivot-based method for elimination of the unknown parameter β from (34) via $Y_k = y_k$) as

$$G_{y_k}(y_l | y_k) = 1 - \frac{m!}{(m-l)!(l-k-1)!(k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \frac{1}{\frac{y_l - y_k}{y_k} (m-l+1+j) + m-k+1+i}. \quad (36)$$

Using the identity

$$\sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \frac{1}{\frac{y_l - y_k}{y_k} (m-l+1+j) + m-k+1+i} = \frac{(k-1)!}{\prod_{i=0}^{k-1} \left(\frac{y_l - y_k}{y_k} (m-l+1+j) + m-k+1+i \right)}, \quad (37)$$

it follows from (36) that

$$G_{y_k}(y_l | y_k) = 1 - \frac{m!}{(m-l)!(l-k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \left[\prod_{i=0}^{k-1} \left(\frac{y_l - y_k}{y_k} (m-l+1+j) + m-k+1+i \right) \right]^{-1} \quad (38)$$

and

$$\bar{G}_{y_k}(y_l | y_k) = 1 - G_{y_k}(y_l | y_k) = \frac{m!}{(m-l)!(l-k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \left[\prod_{i=0}^{k-1} \left(\frac{y_l - y_k}{y_k} (m-l+1+j) + m-k+1+i \right) \right]^{-1}. \quad (39)$$

It follows from (36) that the cumulative distribution function of the ancillary statistic W_{lk}^* is given by

$$G(w_{lk}^*) = 1 - \frac{m!}{(m-l)!(l-k-1)!(k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \frac{1}{w_{lk}^* (m-l+1+j) + m-k+1+i}. \quad (40)$$

The probability density function of the ancillary statistic W_{lk}^* is given by

$$g(w_{lk}^*) = \frac{dG(w_{lk}^*)}{dw_{lk}^*} = \frac{m!}{(m-l)!(l-k-1)!(k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{[w_{lk}^* (m-l+1+j) + m-k+1+i]^2}, \quad w_{lk}^* \in (0, \infty). \quad (41)$$

Using the identity (37), it follows from (40) that

$$G(w_{lk}^*) = 1 - \frac{m!}{(m-l)!(l-k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \left[\prod_{i=0}^{k-1} (w_{lk}^* (m-l+1+j) + m-k+1+i) \right]^{-1} \quad (42)$$

and

$$\bar{G}(w_{lk}^*) = 1 - G(w_{lk}^*) = \frac{m!}{(m-l)!(l-k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \left[\prod_{i=0}^{k-1} (w_{lk}^* (m-l+1+j) + m-k+1+i) \right]^{-1}. \quad (43)$$

8. Constructing Exact Pivot-Based Prediction Limits

The following theorems illustrate the pivot-based approach to constructing exact within-sample prediction limits.

Theorem 2 (Exact lower statistical within-sample $(1-\alpha)$ -prediction limit based on Y_k and S_k). Let $Y_1 \leq \dots \leq Y_k$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution with the probability density function (1) and cumulative distribution function (2). It is assumed that the parameter β is unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and Y_k is a random variable. The exact lower statistical within-sample prediction limit with expected $(1-\alpha)$ -confidence, $L_l^* \equiv L_l^*(Y_k, S_k)$, on future outcomes of the l th order statistic Y_l from a set of m future ordered observations $Y_{k+1} \leq \dots \leq Y_m$ also from the distribution (1), which satisfies

$$E_\beta \left\{ \Pr(Y_l > L_l^* | Y_k = y_k) \right\} = E_\beta \left\{ \int_{L_l^*}^{\infty} g_\beta(y_l | y_k) dy_l \right\} = E_\beta \left\{ \bar{G}_\beta(L_l^* | y_k) \right\} = \bar{G}_{s_k}(L_l^* | y_k) = 1 - \alpha, \quad (44)$$

is given by

$$L_l^* = Y_k + \mathcal{G}_{1-\alpha} S_k, \quad (45)$$

where

$$\mathcal{G}_{1-\alpha} = \arg \left(\frac{\sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j}}{B(l-k, m-l+1)} [1 + (m-l+1+j)\mathcal{G}_{1-\alpha}]^{-k} = 1-\alpha \right). \quad (46)$$

Proof. Assuming that $y_l = L_l^*$, it follows from (31) and (44) that

$$\bar{G}_{s_k}(L_l^* | y_k) = \frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} [1 + (m-l+1+j)\mathcal{G}_{1-\alpha}]^{-k} = 1-\alpha, \quad (47)$$

where

$$\mathcal{G}_{1-\alpha} = \frac{L_l^* - Y_k}{S_k}. \quad (48)$$

Then (45) follows from (48). This ends the proof.

Corollary 2.1. The exact upper statistical within-sample $(1-\alpha)$ - prediction limit U_l^* on future outcomes of the l th order statistic Y_l from a set of m future ordered observations $Y_{k+1} \leq \dots \leq Y_m$ also from the distribution (1), which satisfies

$$E_\beta \left\{ \Pr(Y_l \leq U_l^* | Y_k = y_k) \right\} = E_\beta \left\{ \int_0^{U_l^*} g_\beta(y_l | y_k) dy_l \right\} = E_\beta \left\{ G_\beta(U_l^* | y_k) \right\} = G_{s_k}(U_l^* | y_k) = 1-\alpha, \quad (49)$$

is given by

$$U_l^* = Y_k + \mathcal{G}_\alpha S_k, \quad (50)$$

where $\mathcal{G}_\alpha$ is determined by

$$\mathcal{G}_\alpha = \arg \left(\frac{1}{B(l-k, m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} [1 + (m-l+1+j)\mathcal{G}_\alpha]^{-k} = \alpha \right). \quad (51)$$

Theorem 3 (Exact lower statistical within-sample $(1-\alpha)$ - prediction limit based on Y_k). Let us assume that the first $k-1$ ordered observations (order statistics) $Y_1 \leq \dots \leq Y_{k-1}$ in a sample of size m from the exponential distribution with the probability density function (1) are unknown. Then the exact lower statistical within-sample prediction limit with expected $(1-\alpha)$ -confidence, $L_l^* \equiv L_l^*(Y_k)$, on future outcomes of the l th order statistic Y_l from a set of m future ordered observations $Y_{k+1} \leq \dots \leq Y_m$ also from the distribution (1), which satisfies (44), is given by

$$L_l^* = (1 + \mathcal{G}_{1-\alpha})Y_k, \quad (52)$$

where $\mathcal{G}_{1-\alpha}$ is determined by

$$\mathcal{G}_{1-\alpha} = \arg \left(\frac{m!}{(m-l)!(l-k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \left[\prod_{i=0}^{k-1} (\mathcal{G}_{1-\alpha}(m-l+1+j) + m-k+1+i) \right]^{-1} = 1-\alpha \right). \quad (53)$$

Proof. Assuming that $y_l = L_l^*$, it follows from (42) and (49) that

$$\bar{G}_{y_k}(L_l^* | y_k) = \frac{m!}{(m-l)!(l-k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \times \left[\prod_{i=0}^{k-1} (\mathcal{G}_{1-\alpha}(m-l+1+j) + m-k+1+i) \right]^{-1} = 1-\alpha. \quad (54)$$

where

$$\mathcal{G}_{1-\alpha} = \frac{L_l^* - Y_k}{Y_k}. \quad (55)$$

Then (52) follows from (55). This ends the proof.

Corollary 3.1. The exact upper statistical within-sample $(1-\alpha)$ - prediction limit $U_l^\bullet$ on future outcomes of the l th order statistic Y_l from a set of m future ordered observations $Y_{k+1} \leq \dots \leq Y_m$ also from the distribution (1), which satisfies (49), is given by

$$U_l^\bullet = (1 + \mathcal{G}_\alpha)Y_k, \quad (56)$$

where $\mathcal{G}_\alpha$ is determined by

$$\mathcal{G}_\alpha = \arg \left(\frac{m!}{(m-l)!(l-k-1)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \left[\prod_{i=0}^{k-1} (\mathcal{G}_\alpha(m-l+1+j) + m-k+1+i) \right]^{-1} = \alpha \right). \quad (57)$$

9. Constructing Shortest Length and/or Equal Tails Confidence Intervals for Future Outcomes

Let us assume that $Y_1 \leq \dots \leq Y_k$ are the first k ordered observations from the previous random sample of size m from the exponential distribution (1) (the Type II censored sample and the parameter β is unknown (nuisance)). What is the $100(1-\alpha)\%$ shortest-length confidence interval for a single future observation from a previously sampled population/process (motivated by a customer's request for an interval to contain the life of a purchased product)?

Answer: To illustrate new-sample prediction via the shortest-length confidence interval, we show how to construct the $100(1-\alpha)\%$ shortest-length confidence interval for a single future observation Y from the same distribution (1).

The random variable (ancillary statistic) $W = Y/S_k$ has the cumulative distribution function (22) and probability density function (23). Using W , we can construct a $100(1-\alpha)\%$ confidence interval for Y from

$$P(w_1 \leq W \leq w_2) = P\left(w_1 \leq \frac{Y}{S_k} \leq w_2\right) = P(w_1 S_k \leq Y \leq w_2 S_k) = 1 - \alpha \quad (58)$$

by suitably choosing the decision variables w_1 and w_2 . Hence, the confidence interval for Y is given by

$$[w_1, w_2]. \quad (59)$$

The length of the statistical confidence interval is given by

$$L(w_1, w_2 | S_k) = S_k(w_2 - w_1) \propto L(w_1, w_2) = \int_{w_1}^{w_2} \zeta(\tau) d\tau = \int_{w_1}^{w_2} \tau^0 d\tau = w_2 - w_1. \quad (60)$$

In order to find the shortest length confidence interval $L(w_1, w_2)$, we should find a pair of decision variables w_1 and w_2 such that $L(w_1, w_2)$ is minimum.

Classical analytical approach: the problem statement in terms of the ancillary statistic W and decision variables w_1 and w_2 :

Minimize

$$L(w_1, w_2) = w_2 - w_1 \quad (61)$$

subject to

$$\int_{w_1}^{w_2} f(w) dw = \int_{w_1}^{w_2} \frac{k}{(1+w)^{k+1}} dw = F(w_2) - F(w_1) = \frac{1}{(1+w_1)^k} - \frac{1}{(1+w_2)^k} = 1 - \alpha. \quad (62)$$

Differentiating $L(w_1, w_2)$ with respect to w_1 , we get

$$\frac{dL(w_1, w_2)}{dw_1} = \frac{dw_2}{dw_1} - 1. \quad (63)$$

From (62) we find the derivative of w_2 with respect to w_1 as follows:

$$\frac{d}{dw_1} \int_{w_1}^{w_2} f(w) dw = \frac{d}{dw_1} (1 - \alpha) \quad (64)$$

that is

$$f(w_2) \frac{dw_2}{dw_1} - f(w_1) = 0. \quad (65)$$

Thus, we have

$$\frac{dw_2}{dw_1} = \frac{f(w_1)}{f(w_2)}. \quad (66)$$

Letting this into (63), we get

$$\frac{dL(w_1, w_2)}{dw_1} = \frac{f(w_1)}{f(w_2)} - 1 = \frac{(1 + w_2)^{k+1}}{(1 + w_1)^{k+1}} - 1 > 0. \quad (67)$$

Optimal analytical results. It follows from (62) that

$$1 - \alpha < \frac{1}{(1 + w_1)^k} \leq 1 \left(\text{or } 0 \leq w_1 < \left(\frac{1}{1 - \alpha} \right)^{1/k} - 1 \right), \quad (68)$$

so that the minimum occurs at

$$w_1 = 0, \quad w_2 = \left(\frac{1}{\alpha} \right)^{1/k} - 1 = 1.154435 \text{ (if } \alpha = 0.1, k = 3). \quad (69)$$

The statistical shortest-length confidence interval for Y based on S_k is

$$L(w_1, w_2 | S_k) = S_k L(w_1, w_2) = S_k \left[\left(\frac{1}{\alpha} \right)^{1/k} - 1 \right]. \quad (70)$$

Note that the expected shortest-length confidence interval for Y is given by

$$E_\beta \{L(w_1, w_2 | S_k)\} = E_\beta \{S_k\} \left[\left(\frac{1}{\alpha} \right)^{1/k} - 1 \right], \quad (71)$$

which is minimized, subject to (62).

The expected length of the interval that minimizes $E_\beta \{L(w_1, w_2 | S_k)\}$ is which is also the expected length of the statistical shortest-length confidence interval $L(w_1, w_2 | S_k)$ based on S_k . Note that the length of the interval $\left[0, \left(\frac{1}{\alpha} \right)^{1/k} - 1 \right]$ goes to 0 as $k \rightarrow \infty$.

New simple computational method: the problem statement in terms of the decision variable p (probability) and quantile functions w_1 and w_2 :

Minimize

$$L^2(w_1, w_2) = \left(\int_{w_1}^{w_2} \zeta(\tau) d\tau \right)^2 = \left(\int_{w_1}^{w_2} \tau^0 d\tau \right)^2 = (w_2 - w_1)^2 = \left(F^{-1}(1 - \alpha + p) - F^{-1}(p) \right)^2 = \left[\left(\frac{1}{\alpha - p} \right)^{1/k} - \left(\frac{1}{1 - p} \right)^{1/k} \right]^2, \quad (72)$$

where the p -quantile function w_1 of W is given by

$$w_1 = F^{-1}(p) = \left(\frac{1}{1 - p} \right)^{1/k} - 1 \quad (73)$$

and the $(1 - \alpha + p)$ -quantile function w_2 of W is given by

$$w_2 = F^{-1}(1 - \alpha + p) = \left(\frac{1}{\alpha - p} \right)^{1/k} - 1, \quad (74)$$

subject to

$$0 \leq p \leq \alpha, \quad (75)$$

Numerical solutions. The optimal numerical solution minimizing $L(w_1, w_2)$ can be obtained using the computer software “Solver”. If, for example, $k = 3$, $\alpha = 0.1$, then the optimal numerical solution is given by

$$p = 0, \quad w_1 = 0, \quad w_2 = 1.154435 \quad (76)$$

with the $100(1 - \alpha)\%$ shortest-length confidence interval

$$L(w_1, w_2) = 1.154435. \quad (77)$$

The $100(1 - \alpha)\%$ equal tails confidence interval is given by

$$L(w_1, w_2 \mid p = \alpha / 2) = 1.697174. \quad (78)$$

with

$$p = 0.05, \quad w_1 = 0.017245, \quad w_2 = 1.714418. \quad (79)$$

Inference. The proposed new simple computational method correctly recognized the adequate version of possible decision making and gave accurate numerical results.

Relative efficiency. The relative efficiency of $L(w_1, w_2 \mid p = \alpha / 2)$ as compared with $L(w_1, w_2)$ is given by

$$\text{rel. eff.}_L \{ L(w_1, w_2 \mid p = \alpha / 2), L(w_1, w_2) \} = \frac{L(w_1, w_2)}{L(w_1, w_2 \mid p = \alpha / 2)} = \frac{1.154435}{1.697174} = 0.68021. \quad (80)$$

10. Cost-Effective Planning Unbiased Inspection Strategies

In this paper we look at inspection strategies for items or structures that can be described as being in one of two states, one of which is preferable to the other. The inspection strategy defined is based on the conditional reliability of the structure. It is given as follows. Fix $0 < \gamma < 1$ and let

$$\tau_1 = \arg (\Pr\{Y > \tau_1\} = \gamma), \quad (81)$$

$$\tau_j = \arg (\Pr\{Y > \tau_j \mid Y > \tau_{j-1}\} = \gamma), \quad j \geq 2, \quad (82)$$

where $\{\tau_j\}_{j=1, 2, \dots}$ are inspection times, Y is a random variable representing the lifetime of the component (structure). This is named as ‘reliability-based inspection’. The value of γ can be seen as ‘minimum reliability required’ (or ‘reliability index’). If F_β , the structure lifetime distribution with the parameter β is exponential (1), the definition of the inspection strategy (when the parameter β is unknown) is equivalent to

$$\tau_j = \arg (\bar{F}_{s_k}(\tau_j) = \gamma^j), \quad j \geq 1, \quad (83)$$

where

$$\bar{F}_{s_k}(\tau_j) = 1 - F_{s_k}(\tau_j). \quad (84)$$

If it is known that each inspection costs c_1 and the cost of leaving an undetected failure (some sort of ‘damage’) is c_2 per unit time, then the total expected cost per inspection cycle is given by

$$C(\tau(s_k, \gamma)) = \sum_{j=1}^{\infty} \int_{\tau_{j-1}}^{\tau_j} [jc_1 + c_2(\tau_j - y)] f_{s_k}(y) dy = c_1 \sum_{j=1}^{\infty} j [F_{s_k}(\tau_j) - F_{s_k}(\tau_{j-1})] + c_2 \sum_{j=1}^{\infty} \tau_j [F_{s_k}(\tau_j) - F_{s_k}(\tau_{j-1})] - c_2 \int_0^{\infty} y f_{s_k}(y) dy$$

$$= c_1 \sum_{j=0}^{\infty} \bar{F}_{s_k}(\tau_j) + c_2 \sum_{j=1}^{\infty} \tau_j [\bar{F}_{s_k}(\tau_{j-1}) - \bar{F}_{s_k}(\tau_j)] - c_2 E_{s_k}\{Y\} = c_1 \left[\frac{1}{1-\gamma} + s_k \frac{c_2}{c_1} \left(\frac{1-\gamma^{1/k}}{\gamma^{1/k}-\gamma} - \frac{1}{k-1} \right) \right]. \quad (85)$$

If, say, $c_1=1$, $c_2=2$, $s_k=50$, $k=3$, it follows from (85) that the reliability index γ minimizing the total estimated cost per inspection cycle is given by

$$\gamma^* = \arg \min_{0 < \gamma < 1} C(\tau(s_k, \gamma)) = 0.825776, \quad (86)$$

where

$$C(\tau(s_k, \gamma^*)) = 10.73406. \quad (87)$$

The index of improvement percentage in effectiveness of the optimal inspection strategy (with $\gamma = \gamma^* = 0.825776$) as compared with the standard inspection strategy (with $\gamma = \gamma_{st} = 0.95$) is given by

$$I_{\text{imp.per.}}(\gamma^*, \gamma_{st}) = \frac{C(\tau(s_k, \gamma_{st})) - C(\tau(s_k, \gamma^*))}{C(\tau(s_k, \gamma_{st}))} 100\% = \frac{21.29704 - 10.73406}{21.29704} 100\% = 49.59835\% \cong 50\%. \quad (88)$$

11. Conclusion

The new intelligent computational technique proposed in this paper represents the conceptually simple, efficient and useful method for constructing exact, optimal or improved statistical decision rules under parametric uncertainty of underlying models. This technique is based on the constructive use of the invariance principle in mathematical statistics. We have illustrated the technique for the exponential distribution. Applications to other log-location-scale distributions could follow directly. The methodology described here can be extended in several different directions to solve various problems arising in practice.

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