A New Unified Computational Approach to Intelligent Constructing Shortest-Length or Equal Tails Confidence Intervals under Parametric Uncertainty

N. A. Nechval^{1,*}, G. Berzins¹, K. N. Nechval², Zh. Tsaurkubule³

¹BVEF Research Institute, University of Latvia, Riga, 1586, Latvia.

²Transport and Telecommunication Institute, Riga, 1019, Latvia.

³Baltic International Academy, Riga, 1019, Latvia.

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***Corresponding author:** N. A. Nechval, BVEF Research Institute, University of Latvia, Riga, 1586, Latvia.
Email: nechval@junik.lv

Abstract

A confidence interval is a range of values that gives the user a sense of how precisely a statistic estimates a parameter. In the present paper, a novel unified computational approach is proposed to construct shortest-length or equal tails confidence intervals in terms of pivotal quantities and quantile functions. In statistics, a pivotal quantity (or pivot) is a function of observations and unobservable parameters such that the function's probability distribution does not depend on the unknown parameters (including nuisance parameters). This approach represents a simple and computationally attractive numerical method for finding the shortest-length or equal tails confidence intervals using the pivotal quantities, which are developed from either maximum likelihood estimates or sufficient statistics. Finding a pivotal quantity is not discussed, but the choice a "good" pivotal quantity is essential for the resulting confidence interval to be useful. The unified computational approach yields intervals in several situations which have previously required separate analyses using more advanced techniques and tables for numerical solutions. Unlike the Bayesian approach, the proposed approach is independent of the choice of priors and represents a novelty in the theory of statistical decisions. It allows one to eliminate nuisance parameters from the problem via the technique of invariant statistical embedding and averaging in terms of pivotal quantities (ISE&APQ). To illustrate the proposed approach, numerical examples are given. In detail, the Pareto distribution is discussed.

Keywords

Pivotal quantity, isolating unknown (nuisance) parameters, quantile function, shortest-length confidence interval, equal tails confidence interval

1. Introduction

In many problems of statistical inference, the experimenter is interested in constructing a confidence interval that contains the true (unknown) value of parameter (say, λ) with a specified (high) probability [1]. Confidence intervals allow us to generalize our findings from the samples from which our data was taken to the population from which our survey sample was drawn. This ability to generalize one's findings is often very helpful in social science research. Using a confidence

interval to estimate an unknown parameter is a classical statistical technique that is introduced in almost every statistics course. If we are given a random sample $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$ from a density $f_\lambda(y)$ and a pivotal quantity $V(S(\mathbf{Y}), \lambda)$ (which is developed from either maximum likelihood estimate or sufficient statistic $S(\mathbf{Y})$), whose distribution does not depend on λ , then confidence interval for a single unknown parameter λ is often derived by using a pivotal quantity $V(S(\mathbf{Y}), \lambda)$. To clarify the problem, let $V = V(S(\mathbf{Y}), \lambda)$ be a pivotal quantity with the probability distribution function (PDF) $q(v)$ and cumulative distribution function (CDF) $Q(v)$. Then the problem of constructing shortest-length or equal tails confidence intervals in terms of pivotal quantities can be formulated more generally as follows.

1.1 Classical Approach to Constructing Shortest-Length Confidence Intervals

It is assumed that the length of the statistical confidence interval is given by

$$L(a, b | S(\mathbf{Y})) \propto L(a, b) = \int_a^b \zeta(\tau) d\tau, \quad (1)$$

The length of the expected confidence interval is given by

$$E_\lambda \{L(a, b | S(\mathbf{Y}))\} \propto L(a, b) = \int_a^b \zeta(\tau) d\tau, \quad (2)$$

In order to find the $100(1-\alpha)\%$ statistical (or expected) shortest-length confidence interval for λ , we should find a pair of decision variables a and b such that $L(a, b)$ is minimum.

Problem statement in terms of the pivot V and decision variables a and b :

Minimize

$$L(a, b) = \int_a^b \zeta(\tau) d\tau \quad (3)$$

subject to

$$\int_a^b q(v) dv = Q(b) - Q(a) = 1 - \alpha. \quad (4)$$

Classical analytical approach to solution of the problem:

Differentiating $L(a, b)$ with respect to a , we get

$$\frac{dL(a, b)}{da} = \frac{d}{da} \int_a^b \zeta(\tau) d\tau = \zeta(b) \frac{db}{da} - \zeta(a) \quad (5)$$

From (4) we find the derivative of b with respect to a as follows:

$$\frac{d}{da} \int_a^b q(v) dv = \frac{d}{da} (1 - \alpha) \quad (6)$$

that is

$$q(b) \frac{db}{da} - q(a) = 0. \quad (7)$$

Thus, we have

$$\frac{db}{da} = \frac{q(a)}{q(b)}. \quad (8)$$

Substituting this into (5), we obtain

$$\frac{dL(a, b)}{da} = \zeta(b) \frac{q(a)}{q(b)} - \zeta(a). \quad (9)$$

Now consider the following three versions of possible decision making.

Version 1 of possible decision making:

$$\frac{dL(a,b)}{da} = \zeta(b) \frac{q(a)}{q(b)} - \zeta(a) = 0. \quad (10)$$

Then the optimal analytical solution of the problem is given by

$$\frac{q(a)}{\zeta(a)} = \frac{q(b)}{\zeta(b)}. \quad (11)$$

Version 2 of possible decision making:

$$\frac{dL(a,b)}{da} = \zeta(b) \frac{q(a)}{q(b)} - \zeta(a) < 0. \quad (12)$$

It follows from (4) that

$$1 - \alpha < Q(b) \leq 1. \quad (13)$$

Then the optimal analytical solution of the problem is given by

$$b = \arg[Q(b) = 1], \quad a = \arg[Q(a) = \alpha]. \quad (14)$$

Version 3 of possible decision making:

Let us assume that

$$Q(b) = 1 - \bar{Q}(b), \quad Q(a) = 1 - \bar{Q}(a). \quad (15)$$

It follows from (4) that

$$\int_a^b q(v) dv = Q(b) - Q(a) = \bar{Q}(a) - \bar{Q}(b) = 1 - \alpha. \quad (16)$$

It follows from (16) that

$$\frac{d}{da} \int_a^b q(v) dv = \frac{d}{da} (\bar{Q}(a) - \bar{Q}(b)) = \frac{d}{da} (1 - \alpha) \quad (17)$$

that is

$$\bar{Q}'(a) - \bar{Q}'(b) \frac{db}{da} = 0. \quad (18)$$

Thus, we have

$$\frac{db}{da} = \frac{\bar{Q}'(a)}{\bar{Q}'(b)}. \quad (19)$$

Substituting this into (5), we obtain

$$\frac{dL(a,b)}{da} = \zeta(b) \frac{\bar{Q}'(a)}{\bar{Q}'(b)} - \zeta(a) > 0, \quad (20)$$

It follows from (16) that

$$1 - \alpha < \bar{Q}(a) \leq 1, \quad (21)$$

Then the optimal analytical solution of the problem is given by

$$a = \arg[\bar{Q}(a) = 1], \quad b = \arg[\bar{Q}(b) = \alpha]. \quad (22)$$

1.2 Novel Computational Approach to Constructing Shortest-Length Confidence Intervals

It is assumed that the length of the confidence interval is given by (3). The proposed approach is based on the use of the numerical values of the quantile functions a and b . In order to find the $100(1-\alpha)\%$ shortest-length confidence interval for λ , we should find a pair of numerical values of the quantile functions a and b such that $L(a, b)$ is minimum.

Problem statement in terms of the decision variable p (probability) and quantile functions a and b :

Minimize

$$L^2(a, b) = \left(\int_a^b \zeta(\tau) d\tau \right)^2, \quad (23)$$

where the quantile function a is given by

$$a = Q^{-1}(p), \quad (24)$$

the quantile function b is given by

$$b = Q^{-1}(1 - \alpha + p), \quad (25)$$

subject to

$$0 \leq p \leq \alpha. \quad (26)$$

The decision variable to be determined is p (probability).

Novel computational approach to numerical solution of the problem. The optimal numerical values of the quantile functions a and b , which minimize $L(a, b)$, can be obtained from (23)-(26) using computer software "Solver".

2. Illustrative Examples

2.1 Example 1

Problem statement in terms of the pivot V and decision variables a and b :

Minimize

$$L(a, b) = \int_a^b \zeta(\tau) d\tau = \int_a^b \tau^{-2} d\tau = \frac{1}{a} - \frac{1}{b}. \quad (27)$$

subject to

$$\int_a^b q(v) dv = Q(b) - Q(a) = b^n - a^n = 1 - \alpha, \quad (28)$$

where

$$q(v) = nv^{n-1}, \quad 0 < v < 1, \quad (29)$$

$$Q(u) = \int_0^u q(v) dv = u^n. \quad (30)$$

Classical analytical approach to solution of the problem. It follows from (8) and (27) that

$$\frac{dL(a, b)}{da} = \zeta(b) \frac{d(b)}{d(a)} - \zeta(a) = \zeta(b) \frac{q(a)}{q(b)} - \zeta(a) = b^{-2} \frac{na^{n-1}}{nb^{n-1}} - a^{-2} = \frac{a^{n-1}}{b^{n+1}} - \frac{1}{a^2} = \frac{a^{n+1} - b^{n+1}}{a^2 b^{n+1}} < 0, \quad (31)$$

where $b > a$. It follows from (28) that

$$1 - \alpha < Q(b) \leq 1 \quad (\text{or } (1 - \alpha)^{1/n} < b \leq 1). \quad (32)$$

For this example, version 2 is an adequate version of possible decision making.

Optimal analytical solution. It follows from the above that the minimum of $L(a, b)$ occurs at $b=1$, $a = \alpha^{1/n}$. Thus, the shortest length $(1 - \alpha)$ - confidence interval is given by

$$L(a, b) = \frac{1}{\alpha^{1/n}} - 1. \quad (33)$$

If, say, $n=3$, $\alpha=0.1$, then $L(a, b) = 1.154435$. Note that $L(a, b)$ goes to 0 as $n \rightarrow \infty$.

Problem statement in terms of the decision variable p (probability) and quantile functions a and b .

Minimize

$$L^2(a, b) = \left(\int_a^b \zeta(\tau) d\tau \right)^2 = \left(\int_a^b \tau^{-2} d\tau \right)^2 = \left(\frac{1}{a} - \frac{1}{b} \right)^2 = \left(\frac{1}{p^{1/n}} - \frac{1}{(1-\alpha+p)^{1/n}} \right)^2, \quad (34)$$

where the p -quantile function a of V is given by

$$a = Q^{-1}(p) = p^{1/n} \quad (35)$$

and the $(1-\alpha+p)$ -quantile function b of V is given by

$$b = Q^{-1}(1-\alpha+p) = (1-\alpha+p)^{1/n}, \quad (36)$$

subject to

$$0 \leq p \leq \alpha, \quad (37)$$

Optimal numerical solution. The optimal numerical solution minimizing $L(a, b)$ can be obtained using the computer software “Solver”.

If, for example, $n=3$, $\alpha=0.1$, then the optimal numerical solution is given by

$$p = 0.1, \quad a = 0.464159, \quad b = 1 \quad (38)$$

with the $100(1-\alpha)\%$ shortest-length confidence interval

$$L(a, b) = 1.154435. \quad (39)$$

The $100(1-\alpha)\%$ equal tails confidence interval is given by

$$L(a, b | p = \alpha/2) = 1.697173 \quad (40)$$

with

$$p = 0.05, \quad a = 0.368403, \quad b = 0.983047572. \quad (41)$$

Inference. The proposed approach correctly recognized an adequate version 2 of possible decision making and gave accurate numerical results.

Relative efficiency. The relative efficiency of $L(a, b | p = \alpha/2)$ as compared with $L(a, b)$ is given by

$$\text{rel. eff.}_L \{ L(a, b | p = \alpha/2), L(a, b) \} = \frac{L(a, b)}{L(a, b | p = \alpha/2)} = \frac{1.154435}{1.697173} = 0.680211. \quad (42)$$

2.2 Example 2

Problem statement in terms of the pivot V and decision variables a and b :

Minimize

$$L(a, b) = \int_a^b \zeta(\tau) d\tau = \int_a^b \tau^{-2} d\tau = \frac{1}{a} - \frac{1}{b}. \quad (43)$$

subject to

$$\int_a^b q(v) dv = Q(b) - Q(a) = \bar{Q}(a) - \bar{Q}(b) = \left(\frac{1}{a} \right)^n - \left(\frac{1}{b} \right)^n = 1 - \alpha, \quad (44)$$

where

$$q(v) = n(1/v)^{n+1}, \quad u \geq 1, \quad (45)$$

$$Q(u) = \int_1^u q(v) dv = 1 - (1/u)^n. \quad (46)$$

$$\bar{Q}(u) = 1 - Q(u) = (1/u)^n. \quad (47)$$

Classical analytical approach to solving the problem. It follows from (8) and (43) that

$$\frac{dL(a,b)}{da} = \zeta(b) \frac{d(b)}{d(a)} - \zeta(a) = \zeta(b) \frac{q(a)}{q(b)} - \zeta(a) = b^{-2} \frac{n(1/a)^{n+1}}{n(1/b)^{n+1}} - a^{-2} = \frac{b^{n-1}}{a^{n+1}} - \frac{1}{a^2} = \frac{b^{n-1} - a^{n-1}}{a^{n+1}} > 0, \quad (48)$$

where $b > a$. It follows from (44) that

$$1 - \alpha < \bar{Q}(a) \leq 1 \quad (\text{or } (1 \leq a < 1/(1-\alpha)^{1/n})). \quad (49)$$

For this example, version 3 is an adequate version of possible decision making.

Optimal analytical solution. It follows from the above that the minimum of $L(a,b)$ occurs at $a=1$, $b = 1/\alpha^{1/n}$. Thus,

$$L(a,b) = 1 - \alpha^{1/n}. \quad (50)$$

If, say, $n=3$, $\alpha=0.1$, then $L(a,b) = 0.535841$. Note that $L(a,b)$ goes to 0 as $n \rightarrow \infty$.

Problem statement in terms of the decision variable p (probability) and quantile functions a and b .

Minimize

$$L^2(a,b) = \left(\int_a^b \zeta(\tau) d\tau \right)^2 = \left(\int_a^b \tau^{-2} d\tau \right)^2 = \left(\frac{1}{a} - \frac{1}{b} \right)^2 = \left(\frac{1}{p^{1/n}} - \frac{1}{(1-\alpha+p)^{1/n}} \right)^2, \quad (51)$$

where the p -quantile function a of V is given by

$$a = Q^{-1}(p) = p^{1/n} \quad (52)$$

and the $(1-\alpha+p)$ -quantile function b of V is given by

$$b = Q^{-1}(1-\alpha+p) = (1-\alpha+p)^{1/n}, \quad (53)$$

subject to

$$0 \leq p \leq \alpha, \quad (54)$$

Optimal numerical solution. The optimal numerical solution minimizing $L(a,b)$ can be obtained using the computer software “Solver”. If, for example, $n=3$, $\alpha=0.1$, then the optimal numerical solution is given by

$$p = 0.1, \quad a = 1, \quad b = 2.154435 \quad (55)$$

with the $100(1-\alpha)\%$ shortest-length confidence interval

$$L(a,b) = 1.154435. \quad (56)$$

The $100(1-\alpha)\%$ equal tails confidence interval is given by

$$L(a,b | p = \alpha/2) = 1.697173. \quad (57)$$

with

$$p = 0.05, \quad a = 0.368403, \quad b = 0.983047572. \quad (58)$$

Inference. The proposed approach correctly recognized an adequate version 3 of possible decision making and gave accurate numerical results.

Relative efficiency. The relative efficiency of $L(a,b|p=\alpha/2)$ as compared with $L(a,b)$ is given by

$$\text{rel. eff.}_L \{ L(a,b | p = \alpha/2), L(a,b) \} = \frac{L(a,b)}{L(a,b | p = \alpha/2)} = \frac{1.154435}{1.697173} = 0.680211. \quad (59)$$

2.3 Example 3

Problem statement in terms of the pivot V and decision variables a and b :

Minimize

$$L(a,b) = \int_a^b \zeta(\tau) d\tau = \int_a^b \tau^0 d\tau = b - a. \quad (60)$$

subject to

$$\int_a^b q(v)dv = Q(b) - Q(a) = 1 - \alpha, \quad (61)$$

where

$$q(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right), \quad -\infty < v < \infty, \quad (62)$$

$$Q(u) = \int_{-\infty}^u q(v)dv. \quad (63)$$

Classical analytical approach to solving the problem. It follows from (8) and (60) that

$$\frac{dL(a,b)}{da} = \zeta'(b) \frac{d(b)}{d(a)} - \zeta'(a) = \zeta'(b) \frac{q(a)}{q(b)} - \zeta'(a) = \frac{q(a)}{q(b)} - 1 = 0. \quad (64)$$

It follows from (64) that the minimum of (60) occurs when $q(a) = q(b)$, that is, when

$$a = b \text{ or } a = -b. \quad (65)$$

For this example, version 1 is an adequate version of possible decision making.

Optimal analytical solution. Since $a = b$ does not satisfy

$$\int_a^b q(v)dv = 1 - \alpha, \quad (66)$$

we choose $a = -b$.

Problem statement in terms of the decision variable p (probability) and quantile functions a and b :

Minimize

$$L^2(a,b) = \left(\int_a^b \zeta(\tau) d\tau \right)^2 = \left(\int_a^b \tau^0 d\tau \right)^2 = (b-a)^2, \quad (67)$$

where the p -quantile function a of V is given (via Excel software: $NORM.S.INV(\text{probability } p)$) by

$$a = Q^{-1}(p). \quad (68)$$

and the $(1 - \alpha + p)$ -quantile function b of V is given (via Excel software: $NORM.S.INV(\text{probability } 1 - \alpha + p)$) by

$$b = Q^{-1}(1 - \alpha + p). \quad (69)$$

subject to

$$0 \leq p \leq \alpha. \quad (70)$$

Optimal numerical solution. The optimal numerical solution minimizing $L(a,b)$ can be obtained using the computer software “Solver”. If, for example, $\alpha=0.1$, then the optimal numerical solution is given by

$$p = 0.05, \quad a = -1.64485, \quad b = 1.64485, \quad q(a) = q(b) = 0.103136, \quad (71)$$

with the $100(1-\alpha)\%$ shortest-length confidence interval

$$L(a,b) = 3.289707. \quad (72)$$

The $100(1-\alpha)\%$ equal tails confidence interval is given by

$$L(a,b | p = \alpha / 2) = L(a,b) = 3.289707 \quad (73)$$

with

$$p = 0.05, \quad a = -1.64485, \quad b = 1.64485, \quad q(a) = q(b) = 0.103136. \quad (74)$$

Inference. The proposed approach correctly recognized an adequate version 1 of possible decision making and gave accurate numerical results.

Relative efficiency. The relative efficiency of $L(a,b|p=\alpha/2)$ as compared with $L(a,b)$ is given by

$$\text{rel. eff.}_L \{ L(a, b | p = \alpha / 2), L(a, b) \} = \frac{L(a, b)}{L(a, b | p = \alpha / 2)} = \frac{3.289707}{3.289707} = 1. \quad (75)$$

2.4 Example 4

Problem statement in terms of the pivot V and decision variables a and b :

Minimize

$$L(a, b) = \int_a^b \zeta(\tau) d\tau = \int_a^b \tau^{-2} d\tau = \frac{1}{a} - \frac{1}{b}. \quad (76)$$

subject to

$$\int_a^b q(v) dv = Q(b) - Q(a) = 1 - \alpha, \quad (77)$$

where

$$q(v) = \frac{1}{2^{n/2} \Gamma(n/2)} v^{n/2-1} \exp\left(-\frac{v}{2}\right), \quad v > 0, \quad n > 0, \quad (78)$$

$$Q(u) = \int_{-\infty}^u q(v) dv. \quad (79)$$

Classical approach to solving the problem. It follows from (8) and (76) that

$$\frac{dL(a, b)}{da} = \zeta(b) \frac{d(b)}{d(a)} - \zeta(a) = \frac{1}{b^2} \frac{q(a)}{q(b)} - \frac{1}{a^2}, \quad (80)$$

which vanishes if

$$\frac{1}{b^2} \frac{q(a)}{q(b)} = \frac{1}{a^2}. \quad (81)$$

For this example, version 1 is an adequate version of possible decision making.

Optimal analytical solution. It follows from (81) that the optimal solution is given by

$$a^2 q(a) = b^2 q(b). \quad (82)$$

Numerical results giving values of a and b to four significant places of decimals are available (see Tate and Klett [2]).

Problem statement in terms of the decision variable p (probability) and quantile functions a and b :

Minimize

$$L^2(a, b) = \left(\int_a^b \zeta(\tau) d\tau \right)^2 = \left(\int_a^b \tau^{-2} d\tau \right)^2 = \left(\frac{1}{a} - \frac{1}{b} \right)^2, \quad (83)$$

where the p -quantile function a of V is given (via Excel software: *CHISQ.INV(probability p , deg freedom n)*) by

$$a = Q^{-1}(p). \quad (84)$$

and the $(1 - \alpha + p)$ -quantile function b of V is given (via Excel software: *CHISQ.INV(probability $1 - \alpha + p$, deg freedom n)*) by

$$b = Q^{-1}(1 - \alpha + p). \quad (85)$$

subject to

$$0 \leq p < \alpha. \quad (86)$$

Optimal numerical solution. The optimal numerical solution minimizing $L(a, b)$ can be obtained using the computer software "Solver". If, for example, $n=3$, $\alpha=0.1$, then the optimal numerical solution is given by

$$p = 0.099478, \quad a = 0.58208, \quad b = 17.63810464, \quad q(a) = 0.227512, \quad q(b) = 0.000248 \quad (87)$$

with

$$a^2 q(a) = b^2 q(b) = 0.077. \quad (88)$$

and the $100(1 - \alpha)\%$ shortest-length confidence interval

$$L(a, b) = 1.661282. \quad (89)$$

The $100(1 - \alpha)\%$ equal tails confidence interval is given by

$$L(a, b | p = \alpha / 2) = L(a, b) = 2.714186 \quad (90)$$

with

$$p = 0.05, \quad a = 0.351846, \quad b = 7.814728, \quad q(a) = 0.198465, \quad q(b) = 0.022409. \quad (91)$$

Inference. The proposed approach correctly recognized an adequate version 1 of possible decision making and gave accurate numerical results.

Relative efficiency. The relative efficiency of $L(a, b | p = \alpha/2)$ as compared with $L(a, b)$ is given by

$$\text{rel. eff.}_L \{ L(a, b | p = \alpha/2), L(a, b) \} = \frac{L(a, b)}{L(a, b | p = \alpha/2)} = \frac{1.661282}{2.714186} = 0.612074. \quad (92)$$

3. Pareto Distribution

The Pareto distribution is a power law probability distribution that is used in modeling many observable phenomena in description of social, scientific, geophysical, and actuarial science. It has been widely applied in the area of economics, trade, insurance, meteorology, business and social sciences to model some real life phenomena. There is a wide range of situation where the choice of the model can be made focusing just on the specific task to be accomplished while inaccuracy in unimportant areas is wittingly accepted. Nevertheless, sometimes the challenge is not on the choice of the model but the method of estimation or fitting the model. Such cases may arise in the Pareto distribution when the data is too heavy-tailed or having outliers or extreme values or when the sample sizes are small. The challenge in such cases is not just to choose a model but to find a robust and most appropriate method of fitting the distribution to the data. The Pareto distribution as a model for long tailed distributions has resulted in applications to a number of diverse areas including physics, hydrology, and seismology. It has also found suitable applications in engineering, such as internet telegraphic modeling. The distribution has also found application in sonar and radar as a model of sea clutter returns.

Thus, application of the Pareto distribution covers a wide spectrum of areas ranging from actuarial science, economics, finance to biosciences, telecommunications, and extreme value theory.

The probability density function (pdf) of the two-parameter Pareto distribution (also called the Type I Pareto distribution) is given by

$$f_{\theta}(y) = \frac{\beta^{\delta} \delta}{y^{\delta+1}}, \quad y \geq \beta, \quad \beta, \delta > 0, \quad (93)$$

and the cumulative distribution function (cdf) is given by

$$F_{\theta}(y) = 1 - \left(\frac{\beta}{y} \right)^{\delta}, \quad y \geq \beta, \quad (94)$$

where $\theta = (\beta, \delta)$, β is the scale parameter, and δ is the shape parameter.

Let $Y_1 \leq \dots \leq Y_r$ be the first r ordered observations from the random sample of size n from the Pareto distribution (Type II censored sample). The Type II censored samples, where only the smallest or largest r observations are observed, are common in many applications arising in life testing and animal experimentation. For example, this type of censoring occurs when n items are put on test until the first events or failure times are observed and the experiment is stopped. The Type II censored sample may be singly censored or doubly censored, with both the m_1 largest and m_2 smallest observations censored.

3.1 Likelihood Function of the Type II Censored Sample

To summarize the data's evidence about unknown parameters, one of the most fundamental concepts of modern statistics is that of likelihood. The likelihood function for the Type II censored sample is given by

$$\begin{aligned} L(\theta | y) &= \prod_{i=1}^r f_{\theta}(y_i) [1 - F_{\theta}(y_r)]^{n-r} = \prod_{i=1}^r \frac{\beta^{\delta} \delta}{y_i^{\delta+1}} \left[\left(\frac{\beta}{y_r} \right)^{\delta} \right]^{n-r} \\ &= \beta^{r\delta} \delta^r \prod_{i=1}^r y_i^{-(\delta+1)} \left(\frac{\beta}{y_r} \right)^{(n-r)\delta} = \beta^{n\delta} \delta^r \left(\prod_{i=1}^r y_i y_r^{(n-r)} \right)^{-\delta} \left(\prod_{i=1}^r y_i \right)^{-1}. \end{aligned} \quad (95)$$

For computational convenience, the natural log of the likelihood function is determined:

$$l(\theta | y) = \ln L(\theta | y) = n\delta \ln \beta + r \ln \delta - \delta \left(\sum_{i=1}^r \ln y_i + (n-r) \ln y_r \right) - \sum_{i=1}^r \ln y_i. \quad (96)$$

To obtain the maximum likelihood estimate (MLE) of each parameter and an adequate probability distribution of pivotal

quantity (or simply pivot), we consider the following cases of parametric uncertainty: 1) *the scale parameter β is unknown*, 2) *the shape parameter δ is unknown*, 3) *the scale parameter β and shape parameter δ are unknown*.

3.2 Finding the MLE of β and Adequate Probability Distribution of Pivotal Quantity

When the scale parameter β is unknown (but shape parameter δ is known), we may maximize $L(\theta | y)$ subject to the constraint:

$$\beta \leq Y_{(1)}, \quad (97)$$

where $Y_{(1)} = \min_i Y_i$. Thus, we see that L is maximized w.r.t. β when

$$\beta = Y_{(1)}. \quad (98)$$

Therefore, β is estimated by the minimum value of the observation. It follows from (93) and (94) that the probability density function (PDF) of $Y_{(1)}$ is given by

$$q_\beta(y_{(1)} | \delta) = n[1 - F_\theta(y_{(1)})]^{n-1} f_\theta(y_{(1)}) = \frac{n\delta}{y_{(1)}^{n\delta+1}} \beta^{n\delta}, \quad y_{(1)} \geq \beta, \quad (99)$$

and cumulative distribution function (CDF) is given by

$$Q_\beta(y_{(1)} | \delta) = 1 - [1 - F_\theta(y_{(1)})]^n = 1 - \left[1 - 1 + \left(\frac{\beta}{y_{(1)}}\right)^\delta\right]^n = 1 - \left(\frac{\beta}{y_{(1)}}\right)^{n\delta}, \quad y_{(1)} \geq \beta. \quad (100)$$

It follows from (100) that the random variable

$$V = \frac{Y_{(1)}}{\beta} \quad (101)$$

represents a pivotal quantity with the cumulative distribution function

$$Q(v | \delta) = 1 - \left(\frac{1}{v}\right)^{n\delta}, \quad v \geq 1, \quad (102)$$

and probability density function

$$q(v | \delta) = Q'_\delta(v | \delta) = n\delta \left(\frac{1}{v}\right)^{n\delta+1}, \quad v \geq 1, \quad (103)$$

The p -quantile function $Q^{-1}(p | \delta)$ of V is given by

$$Q^{-1}(p | \delta) = \frac{1}{(1-p)^{1/n\delta}}, \quad 0 < p < 1. \quad (104)$$

and the $(1-\alpha+p)$ -quantile function $Q^{-1}(1-\alpha+p | \delta)$ of V is given by

$$Q^{-1}(p | \delta) = \frac{1}{(\alpha-p)^{1/n\delta}}, \quad 0 < p < 1. \quad (105)$$

Thus, if the scale parameter β is unknown, the complete sufficient statistic for this parameter will be $Y_{(1)}$ with the probability density function (99) and cumulative distribution function (100).

3.3 Finding the MLE of δ and Adequate Probability Distribution of Pivotal Quantity

To obtain the estimate $\hat{\delta}$ for δ , we differentiate equation (96) with respect to δ and equate it to zero:

$$\frac{dl(\theta | y)}{d\delta} = n \ln \beta + \frac{r}{\delta} - \sum_{i=1}^r \ln y_i - (n-r) \ln y_r = 0. \quad (106)$$

Simplifying the equation and making δ the subject formula, we have that

$$\hat{\delta} = \frac{r}{\sum_{i=1}^r \ln Y_i + (n-r) \ln Y_r - n \ln \beta} = \frac{r}{\sum_{i=1}^r \ln \left(\frac{Y_i}{\beta}\right) + (n-r) \ln \left(\frac{Y_r}{\beta}\right)} = \frac{r}{T}. \quad (107)$$

where

$$T = \sum_{i=1}^r \ln \left(\frac{Y_i}{\beta} \right) + (n-r) \ln \left(\frac{Y_r}{\beta} \right). \quad (108)$$

It follows from (95) and (108) that

$$L(\theta | y) = \beta^{n\delta} \delta^r \left(\prod_{i=1}^r y_i y_r^{(n-r)} \right)^{-\delta} \left(\prod_{i=1}^r y_i \right)^{-1} = \delta^r \exp(-\delta t) \left(\prod_{i=1}^r y_i \right)^{-1} \propto \delta^r \exp(-\delta t). \quad (109)$$

It follows from (109) that the probability density function of T is given by

$$h_\delta(t) = \frac{\delta^r t^{r-1} \exp(-\delta t)}{\int_0^\infty \delta^r t^{r-1} \exp(-\delta t) dt} = \frac{\delta^r}{\Gamma(r)} t^{r-1} \exp(-\delta t), \quad t \geq 0, \quad \delta > 0. \quad (110)$$

The cumulative distribution function of T is given by

$$H_\delta(t) = \int_0^t \frac{\delta^r}{\Gamma(r)} \tau^{r-1} \exp(-\delta \tau) d\tau, \quad \tau \geq 0. \quad (111)$$

It follows from (110) that the random variable

$$W = \delta T \quad (112)$$

represents a pivotal quantity with the probability density function

$$h(w) = \frac{1}{\Gamma(w)} w^{r-1} \exp(-w), \quad w \geq 0, \quad (113)$$

and cumulative distribution function

$$H(w) = \int_0^w \frac{1}{\Gamma(r)} \omega^{r-1} \exp(-\omega) d\omega, \quad \omega \geq 0. \quad (114)$$

Thus, if the shape parameter δ is unknown, the complete sufficient statistic for this parameter will be T (108) with the probability density function (110) and cumulative distribution function (111).

3.4 Finding the MLE of β and δ and Adequate Probability Distributions of Pivotal Quantities

In this case, we have that the MLE of β is given by

$$\hat{\beta} = Y_{(1)} \sim q_\theta(y_{(1)}) = \frac{n\delta}{y_{(1)}^{n\delta+1}} \beta^{n\delta}, \quad y_{(1)} \geq \beta, \quad (115)$$

$$Q_\theta(y_{(1)}) = 1 - \left(\frac{\beta}{y_{(1)}} \right)^{n\delta}, \quad y_{(1)} \geq \beta. \quad (116)$$

$$\frac{Y_{(1)}}{\beta} = V \sim q_\delta(v) = n\delta \left(\frac{1}{v} \right)^{n\delta+1}, \quad v \geq 1, \quad (117)$$

$$Q_\delta(v) = 1 - \left(\frac{1}{v} \right)^{n\delta}, \quad v \geq 1, \quad (118)$$

where δ is the unknown (nuisance) parameter.

It follows from (95) that

$$\begin{aligned} L(\theta | y; \beta = y_{(1)}) &= \prod_{i=2}^r f_\theta(y_i) [1 - F_\theta(y_r)]^{n-1-(r-1)} = \prod_{i=2}^r \frac{y_{(1)}^\delta \delta}{y_i^{n\delta+1}} \left[\left(\frac{y_{(1)}}{y_r} \right)^\delta \right]^{n-1-(r-1)} \\ &= y_{(1)}^{(r-1)\delta} \delta^{r-1} \prod_{i=2}^r y_i^{-(\delta+1)} \left(\frac{y_{(1)}}{y_r} \right)^{[n-1-(r-1)]\delta} = y_{(1)}^{(n-1)\delta} \delta^{r-1} \left(\prod_{i=2}^r y_i y_r^{[n-1-(r-1)]} \right)^{-\delta} \left(\prod_{i=2}^r y_i \right)^{-1}, \end{aligned} \quad (119)$$

$$l(\theta | y) = \ln L(\theta | y) = (n-1)\delta \ln y_{(1)} + (r-1) \ln \delta - \delta \left(\sum_{i=2}^r \ln y_i + [n-1-(r-1)] \ln y_r \right) - \sum_{i=2}^r \ln y_i. \quad (120)$$

To obtain the estimate $\hat{\delta}$ for δ , we differentiate equation (120) with respect to δ and equate it to zero:

$$\frac{dl(\theta | y)}{d\delta} = (n-1) \ln y_{(1)} + \frac{r-1}{\delta} - \sum_{i=2}^r \ln y_i - [n-1-(r-1)] \ln y_r = 0. \quad (121)$$

Simplifying the equation and making δ the subject formula, we have that

$$\hat{\delta} = \frac{r-1}{\sum_{i=2}^r \ln Y_i + [n-1-(r-1)] \ln Y_r - (n-1) \ln Y_{(1)}} = \frac{r-1}{\sum_{i=2}^r \ln \left(\frac{Y_i}{Y_{(1)}} \right) + [n-1-(r-1)] \ln \left(\frac{Y_r}{Y_{(1)}} \right)} = \frac{r-1}{T_{\bullet}}, \quad (122)$$

where

$$T_{\bullet} = \sum_{i=2}^r \ln \left(\frac{Y_i}{Y_{(1)}} \right) + [n-1-(r-1)] \ln \left(\frac{Y_r}{Y_{(1)}} \right) = \sum_{i=1}^r \ln \left(\frac{Y_i}{Y_{(1)}} \right) + (n-r) \ln \left(\frac{Y_r}{Y_{(1)}} \right). \quad (123)$$

It follows from (119) and (123) that

$$L(\theta | y; \beta = y_{(1)}) = y_{(1)}^{(n-1)\delta} \delta^{r-1} \left(\prod_{i=2}^r y_i y_r^{[n-1-(r-1)]} \right)^{-\delta} \left(\prod_{i=2}^r y_i \right)^{-1} = \delta^{r-1} \exp(-\delta t_{\bullet}) \left(\prod_{i=2}^r y_i \right)^{-1} \propto \delta^{r-1} \exp(-\delta t_{\bullet}). \quad (124)$$

It follows from (124) that the probability density function of $T_{\bullet}$ is given by

$$h_{\delta}(t_{\bullet}) = \frac{\delta^{r-1} t_{\bullet}^{r-2} \exp(-\delta t_{\bullet})}{\int_0^{\infty} \delta^{r-1} t_{\bullet}^{r-2} \exp(-\delta t_{\bullet}) dt_{\bullet}} = \frac{\delta^{r-1}}{\Gamma(r-1)} t_{\bullet}^{r-2} \exp(-\delta t_{\bullet}), \quad t_{\bullet} \geq 0, \quad \delta > 0. \quad (125)$$

The cumulative distribution function of $T_{\bullet}$ is given by

$$H_{\delta}(t_{\bullet}) = \int_0^{t_{\bullet}} \frac{\delta^{r-1}}{\Gamma(r-1)} \tau^{r-2} \exp(-\delta \tau) d\tau, \quad \tau \geq 0. \quad (126)$$

It follows from (126) that the random variable

$$W_{\bullet} = \delta T_{\bullet} \quad (127)$$

represents a pivotal quantity with the probability density function

$$h(w_{\bullet}) = \frac{1}{\Gamma(r-1)} w_{\bullet}^{r-2} \exp(-w_{\bullet}), \quad w_{\bullet} \geq 0, \quad (128)$$

and cumulative distribution function

$$H(w_{\bullet}) = \int_0^{w_{\bullet}} \frac{1}{\Gamma(r-1)} \omega^{r-2} \exp(-\omega) d\omega, \quad \omega \geq 0. \quad (129)$$

Thus, if the scale parameter β and shape parameter δ are unknown, the complete sufficient statistic for the parameter $\theta = (\beta, \delta)$ is given by

$$S = \left(Y_{(1)} = \min Y_i, \quad T_{\bullet} = \sum_{i=1}^r \ln \frac{Y_i}{Y_{(1)}} + (n-r) \ln \frac{Y_r}{Y_{(1)}} \right), \quad (130)$$

where

$$S = (Y_{(1)}, T_{\bullet}) \sim q_{\theta}(y_{(1)}) h_{\delta}(t_{\bullet}) = \frac{n\delta}{y_{(1)}^{n\delta+1}} \beta^{n\delta} \frac{\delta^{r-1}}{\Gamma(r-1)} t_{\bullet}^{r-2} \exp(-\delta t_{\bullet}), \quad y_{(1)} \geq \beta, \quad t_{\bullet} \geq 0, \quad \beta > 0, \quad \delta > 0. \quad (131)$$

In order to exclude the unknown (nuisance) parameter δ from (116), we use the technique of invariant statistical embedding and averaging in terms of pivotal quantities (ISE&APQ) [3-12] as follows. We reduce (116) to

$$Q_{\theta}(y_{(1)}) = 1 - \left(\frac{\beta}{y_{(1)}} \right)^{\frac{n\delta t_{\bullet}}{t_{\bullet}}} = 1 - \left(\frac{\beta}{y_{(1)}} \right)^{\frac{nw_{\bullet}}{t_{\bullet}}} = 1 - \exp \left[\ln \left(\frac{\beta}{y_{(1)}} \right) \frac{nw_{\bullet}}{t_{\bullet}} \right] = 1 - \exp \left[\frac{nw_{\bullet}}{t_{\bullet}} \ln \left(\frac{\beta}{y_{(1)}} \right) \right] = Q_{\beta}(y_{(1)} | w_{\bullet}), \quad y_{(1)} \geq \beta. \quad (132)$$

Then (132) can be transformed to cumulative distribution function (CDF) of $Y_{(1)}$ as follows:

$$\begin{aligned} Q_{\beta}(y_{(1)}) &= E\{Q_{\beta}(y_{(1)} | w_{\bullet})\} = \int_0^{\infty} Q_{\beta}(y_{(1)} | w_{\bullet}) h(w_{\bullet}) dw_{\bullet} = \int_0^{\infty} \left(1 - \exp\left[\frac{nw_{\bullet}}{t_{\bullet}} \ln\left(\frac{\beta}{y_{(1)}}\right)\right]\right) \frac{1}{\Gamma(r-1)} w_{\bullet}^{r-2} \exp(-w_{\bullet}) dw_{\bullet} \\ &= 1 - \frac{1}{\left[1 - \frac{n}{t_{\bullet}} \ln\left(\frac{\beta}{y_{(1)}}\right)\right]^{r-1}} = 1 - \frac{1}{\left[1 + \frac{n}{t_{\bullet}} \ln\left(\frac{y_{(1)}}{\beta}\right)\right]^{r-1}}, \quad y_{(1)} \geq \beta, \end{aligned} \quad (133)$$

where the unknown(nuisance) parameter δ is eliminated from the problem. In this case, the probability density function (PDF) of $Y_{(1)}$ is given by

$$q_{\beta}(y_{(1)}) = Q'_{\beta}(y_{(1)}) = (r-1) \left[1 + \frac{n}{t_{\bullet}} \ln\left(\frac{y_{(1)}}{\beta}\right)\right]^{-r} \frac{n}{t_{\bullet}} \frac{1}{y_{(1)}}, \quad y_{(1)} \geq \beta. \quad (134)$$

It follows from (133) that the cumulative distribution function (CDF) of the pivotal quantity $V = Y_{(1)}/\beta$ is given by

$$Q(v) = 1 - \frac{1}{\left[1 + \frac{n}{t_{\bullet}} \ln(v)\right]^{r-1}}, \quad v \geq 1. \quad (135)$$

It follows from (135) that the probability density function (PDF) of V is given by

$$q(v) = Q'(v) = (r-1) \left[1 + \frac{n}{t_{\bullet}} \ln(v)\right]^{-r} \frac{n}{t_{\bullet}} \frac{1}{v}, \quad v \geq 1 \quad (136)$$

3.5 Constructing Shortest-Length Confidence Interval for the Unknown Parameter β

If $Y_1 \leq \dots \leq Y_r$ are the first r ordered observations from the random sample of size n from the Pareto distribution (93) (the Type II censored sample and the parameter δ is unknown (nuisance)), what is the $100(1-\alpha)\%$ shortest-length confidence interval for the unknown parameter β ?

Answer: In this case, $Y_{(1)} = \min_i Y_i$ is a complete sufficient statistic for β with CDF (133) and PDF (134). The random variable (pivot) $V = Y_{(1)} / \beta$ has CDF (135) and PDF (136). Using V as a pivot, we can construct a $100(1-\alpha)\%$ confidence interval for β from

$$P(a \leq V \leq b) = P\left(a \leq \frac{Y_{(1)}}{\beta} \leq b\right) = P\left(\frac{Y_{(1)}}{b} \leq \beta \leq \frac{Y_{(1)}}{a}\right) = 1 - \alpha \quad (137)$$

by suitably choosing the decision variables a and b . Hence, the confidence interval for β is given by

$$\left[\frac{1}{b}, \frac{1}{a}\right]. \quad (138)$$

The length of the statistical confidence interval is given by

$$L(a, b | Y_{(1)}) = Y_{(1)} (1/a - 1/b) \propto L(a, b) = \int_a^b \zeta(\tau) d\tau = \int_a^b \tau^{-2} d\tau = \frac{1}{a} - \frac{1}{b}. \quad (139)$$

In order to find the shortest length confidence interval $L(a, b)$, we should find a pair of decision variables a and b such that $L(a, b)$ is minimum.

Problem statement in terms of the pivot V and decision variables a and b :

Minimize $L(a, b) = 1/a - 1/b$ subject to

$$\int_a^b q(v)dv = \int_a^b (r-1) \left[1 + \frac{n}{t_{\bullet}} \ln(v) \right]^{-r} \frac{n}{t_{\bullet}} \frac{1}{v} dv = Q(b) - Q(a)$$

$$= \left[1 + \frac{n}{t_{\bullet}} \ln(a) \right]^{-(r-1)} - \left[1 + \frac{n}{t_{\bullet}} \ln(b) \right]^{-(r-1)} = 1 - \alpha. \quad (140)$$

Classical approach to solution of the problem. Differentiating $L(a, b)$ with respect to a , we get

$$\frac{dL(a, b)}{da} = Y_{(1)} \left(-\frac{1}{a^2} + \frac{1}{b^2} \frac{db}{da} \right) \quad (141)$$

From (141) we find the derivative of b with respect to a as follows:

$$\frac{d}{da} \int_a^b q(v)dv = \frac{d}{da} (1 - \alpha) \quad (142)$$

that is

$$q(b) \frac{db}{da} - q(a) = 0. \quad (143)$$

Thus, we have

$$\frac{db}{da} = \frac{q(a)}{q(b)}. \quad (144)$$

Letting this into (141), we get

$$\begin{aligned} \frac{dL(a, b)}{da} &= -\frac{1}{a^2} + \frac{1}{b^2} \frac{q(a)}{q(b)} = -\frac{1}{a^2} + \frac{1}{b^2} \frac{(r-1) \left[1 + \frac{n}{t_{\bullet}} \ln(a) \right]^{-r} \frac{n}{t_{\bullet}} \frac{1}{a}}{(r-1) \left[1 + \frac{n}{t_{\bullet}} \ln(b) \right]^{-r} \frac{n}{t_{\bullet}} \frac{1}{b}} \\ &= -\frac{1}{a^2} + \frac{1}{ab} \frac{\left[1 + \frac{n}{t_{\bullet}} \ln(b) \right]^r}{\left[1 + \frac{n}{t_{\bullet}} \ln(a) \right]^r} = \frac{-b \left[1 + \frac{n}{t_{\bullet}} \ln(a) \right]^r + a \left[1 + \frac{n}{t_{\bullet}} \ln(b) \right]^r}{a^2 b \left[1 + \frac{n}{t_{\bullet}} \ln(a) \right]^r} > 0. \end{aligned} \quad (145)$$

Optimal analytical solution. It follows from (140) that

$$1 - \alpha < \left[1 + \frac{n}{t_{\bullet}} \ln(a) \right]^{-(r-1)} \leq 1 \left(\text{or } 1 \leq a < \exp \left(\frac{t_{\bullet}}{n} \left[\left(\frac{1}{1-\alpha} \right)^{\frac{1}{r-1}} - 1 \right] \right) \right), \quad (146)$$

so that the minimum occurs at

$$a = 1, \quad b = \exp \left(\frac{t_{\bullet}}{n} \left[\left(\frac{1}{\alpha} \right)^{\frac{1}{r-1}} - 1 \right] \right). \quad (147)$$

For this example, version 3 is an adequate version of possible decision making.

The statistical shortest-length confidence interval for β based on $Y_{(1)}$ is

$$L(a, b | Y_{(1)}) = Y_{(1)} L(a, b) = Y_{(1)} (1/b, 1) = (Y_{(1)}/b, Y_{(1)}). \quad (148)$$

Note that the expected shortest-length confidence interval for β is given by

$$E_{\beta} \{L(a, b | Y_{(1)})\} = E_{\beta} \{Y_{(1)}\} L(a, b) = E_{\beta} \{Y_{(1)}\} (1/b, 1) = (E_{\beta} \{Y_{(1)}\}/b, E_{\beta} \{Y_{(1)}\}), \quad (149)$$

which is minimized subject to

$$L(a, b) = Q(b) - Q(a) = \left[1 + \frac{n}{t_{\bullet}} \ln(a) \right]^{-(r-1)} - \left[1 + \frac{n}{t_{\bullet}} \ln(b) \right]^{-(r-1)} = 1 - \alpha, \quad (150)$$

where a and b are given by (147). The expected length of the interval that minimizes $E_{\beta} \{L(a, b | Y_{(1)})\}$ is which is also the

expected length of the statistical shortest-length confidence interval $L(a, b | Y_{(1)})$ based on $Y_{(1)}$. Note that the length of the interval $(Y_{(1)} / b, Y_{(1)})$ goes to 0 as $n \rightarrow \infty$.

Problem statement in terms of the decision variable p (probability) and quantile functions a and b :

Minimize

$$L^2(a, b) = \left(\int_a^b \zeta(\tau) d\tau \right)^2 = \left(\int_a^b \tau^{-2} d\tau \right)^2 = \left(\frac{1}{a} - \frac{1}{b} \right)^2 = \left(\frac{1}{Q^{-1}(p)} - \frac{1}{Q^{-1}(1-\alpha+p)} \right)^2, \quad (151)$$

where the p -quantile function a of V is given by

$$a = Q^{-1}(p) = \exp \left(\frac{t_{\bullet}}{n} \left[\left(\frac{1}{1-p} \right)^{\frac{1}{r-1}} - 1 \right] \right) \quad (152)$$

and the $(1-\alpha+p)$ -quantile function b of V is given by

$$b = Q^{-1}(1-\alpha+p) = \exp \left(\frac{t_{\bullet}}{n} \left[\left(\frac{1}{\alpha-p} \right)^{\frac{1}{r-1}} - 1 \right] \right), \quad (153)$$

subject to

$$0 \leq p \leq \alpha, \quad (154)$$

Optimal numerical solution. The optimal numerical solution minimizing $L(a, b)$ can be obtained using the computer software “Solver”. If, for example, $t_{\bullet} = 1$, $r = 3$, $n = 5$, $\alpha = 0.1$, then the optimal numerical solution is given by

$$p = 0, \quad a = 1, \quad b = 1.541037 \quad (155)$$

with the $100(1-\alpha)\%$ shortest-length confidence interval

$$L(a, b) = 0.351086. \quad (156)$$

The $100(1-\alpha)\%$ equal tails confidence interval is given by

$$L(a, b | p = \alpha/2) = 0.495457. \quad (157)$$

with

$$p = 0.05, \quad a = 1.005209, \quad b = 2.002562. \quad (158)$$

Inference. The proposed approach correctly recognized an adequate version 3 of possible decision making and gave accurate numerical results.

Relative efficiency. The relative efficiency of $L(a, b | p = \alpha/2)$ as compared with $L(a, b)$ is given by

$$\text{rel. eff.}_L \{ L(a, b | p = \alpha/2), L(a, b) \} = \frac{L(a, b)}{L(a, b | p = \alpha/2)} = \frac{0.351086}{0.495457} = 0.708610. \quad (159)$$

3.6 Constructing Shortest-Length Confidence Interval for the Unknown Parameter δ

If $Y_1 \leq \dots \leq Y_r$ are the first r ordered observations from the random sample of size n from the Pareto distribution (93) (the Type II censored sample and the parameter β is unknown), what is the $100(1-\alpha)\%$ shortest-length confidence interval for the unknown parameter δ ?

Answer: In this case, the random variable $W_{\bullet} = \delta T_{\bullet}$ has PDF (128) and CDF (129). Using $W_{\bullet}$ as a pivot, we can construct a $100(1-\alpha)\%$ confidence interval for δ from

$$P(a \leq W_{\bullet} \leq b) = P(a \leq \delta T_{\bullet} \leq b) = P\left(\frac{a}{T_{\bullet}} \leq \delta \leq \frac{b}{T_{\bullet}}\right) = 1 - \alpha \quad (160)$$

by suitably choosing the decision variables a and b . Hence, the confidence interval for δ is given by

$$[a, b]. \quad (161)$$

The length of the statistical confidence interval is given by

$$L(a, b | T_{\bullet}) = \frac{1}{T_{\bullet}}(b - a) \propto L(a, b) = \int_a^b \zeta(\tau) d\tau = \int_a^b \tau^0 d\tau = b - a. \quad (162)$$

In order to find the shortest length confidence interval $L(a, b)$, we should find a pair of decision variables a and b such that $L(a, b)$ is minimum.

Problem statement in terms of the pivot W_* and decision variables a and b :

Minimize $L(a, b) = b - a$ subject to

$$\int_a^b h(w_*) dw_* = \int_a^b \frac{1}{\Gamma(w_* - 1)} w_*^{r-2} \exp(-w_*) dw_* = H(b) - H(a) = 1 - \alpha. \quad (163)$$

Classical approach to solving the problem. Differentiating $L(a, b)$ with respect to a , we get

$$\frac{dL(a, b)}{da} = \left(\frac{db}{da} - 1 \right). \quad (164)$$

From (163) we find the derivative of b with respect to a as follows:

$$\frac{d}{da} \int_a^b h(w_*) dw_* = \frac{d}{da} (1 - \alpha) \quad (165)$$

that is

$$h(b) \frac{db}{da} - h(a) = 0. \quad (166)$$

Thus, we have

$$db / da = h(a) / h(b). \quad (167)$$

Letting this into (164), we get

$$\frac{dL(a, b)}{da} = \frac{db}{da} - 1 = \frac{h(a)}{h(b)} - 1, \quad (168)$$

which vanishes if

$$h(a) / h(b) - 1 = 0. \quad (169)$$

For this example, version 1 is an adequate version of possible decision making.

Optimal analytical solution.

$$h(a) = h(b). \quad (170)$$

Problem statement in terms of the decision variable p (probability) and quantile functions a and b :

Minimize

$$L^2(a, b) = \left(\int_a^b \zeta(\tau) d\tau \right)^2 = \left(\int_a^b \tau^0 d\tau \right)^2 = (b - a)^2 = \left(H^{-1}(1 - \alpha + p) - H^{-1}(p) \right)^2, \quad (171)$$

where the p -quantile function a of W_* is given (via Excel software: GAMMAINV(probability p , $r-1$, 1)) by

$$a = H^{-1}(p). \quad (172)$$

and the $(1 - \alpha + p)$ -quantile function b of W_* is given (via Excel software: GAMMAINV(probability $1 - \alpha + p$, $r-1$, 1)) by

$$b = H^{-1}(1 - \alpha + p). \quad (173)$$

subject to

$$0 \leq p < \alpha. \quad (174)$$

Optimal numerical solution. The optimal numerical solution minimizing $L(a, b)$ can be obtained using the computer software "Solver". If, for example, $r = 4$, $\alpha = 0.1$, then the optimal numerical solution is given by

$$p = 0.010328, \quad a = 0.441327, \quad b = 5.479175, \quad h(a) = h(b) = 0.062636. \quad (175)$$

with the $100(1 - \alpha)\%$ shortest-length confidence interval

$$L(a, b) = 5.037848. \quad (176)$$

The $100(1 - \alpha)\%$ equal tails confidence interval is given by

$$L(a, b | p = \alpha / 2) = 5.478102. \quad (177)$$

with

$$p = 0.05, \quad a = 0.817691, \quad b = 6.295794, \quad h(a) = 0.147581, \quad h(b) = 0.036546. \quad (178)$$

Inference. The proposed approach correctly recognized an adequate version 1 of possible decision making and gave accurate numerical results.

Relative efficiency. The relative efficiency of $L(a, b | p = \alpha/2)$ as compared with $L(a, b)$ is given by

$$\text{rel. eff.}_{L\{L(a, b | p = \alpha/2), L(a, b)\}} = \frac{L(a, b)}{L(a, b | p = \alpha/2)} = \frac{5.037848}{5.478102} = 0.919634. \quad (179)$$

4. Conclusion

The novel unified computational approach proposed in this paper represents the conceptually simple, efficient and useful method for constructing exact statistical (or expected) shortest-length or equal tails confidence intervals in terms of pivotal quantities and quantile functions. This technique, using the experimental complete or type II censored data, is based on the idea of invariant embedding of a sample statistic in the underlying model to construct pivotal quantities and to exclude the unknown (nuisance) parameters from the problem via pivotal quantity averaging. In this case, the exact statistical (or expected) shortest-length or equal tails confidence intervals associated with sampling from corresponding distributions can be found easily and quickly making recognition of adequate versions of possible decisions, tables for numerical solutions, more advanced techniques and special computer programs unnecessary. For example, the special computer program for (170) is given below:

Minimize

$$z = (h(a) - h(b))^2 \quad (180)$$

where the p -quantile function a of $W_{\bullet}$ is given (via Excel software: GAMMAINV (probability p , $r-1$, 1)) by

$$a = H^{-1}(p). \quad (181)$$

and the $(1 - \alpha + p)$ -quantile function b of $W_{\bullet}$ is given (via Excel software: GAMMAINV(probability $1 - \alpha + p$, $r-1$, 1)) by

$$b = H^{-1}(1 - \alpha + p). \quad (182)$$

subject to

$$0 \leq p < \alpha. \quad (183)$$

Optimal numerical solution. The optimal numerical solution minimizing z can be obtained using the computer software "Solver". If, for example, $r = 4$, $\alpha = 0.1$, then the optimal numerical solution is given by

$$p = 0.010328, \quad a = 0.441327, \quad b = 5.479175, \quad h(a) = h(b) = 0.062636. \quad (184)$$

with the $100(1 - \alpha)\%$ shortest-length confidence interval

$$L(a, b) = 5.037848. \quad (185)$$

The main advantage of the proposed approach is that it includes only one decision variable (probability p) for making decisions under constraints. In other words, the two decision variables a and b are reduced to one decision variable (probability p). This approach greatly simplifies the problem of constructing shortest-length or equal tails confidence intervals for unknown parameters of various distributions and is a novelty in the theory of statistical decisions regarding confidence intervals.

The unified computational method described in the paper is illustrated in detail for the case of constructing shortest-length or equal tails confidence intervals for the two-parameter Pareto distribution. Applications of this method to construct shortest-length or equal tails confidence intervals for unknown parameters of log-location-scale or other probability distributions can follow directly.

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