Improved High Order Methods Using Boundary Layer Detection for a Singular Perturbation Problem

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Abstract

A singular perturbation problem is solved with improved high order methods using boundary layer detection theorems. The stability and convergence, independent of the singular perturbation parameter, is numerically verified.

Keywords

Singular Perturbation, Differential Equations, Boundary Layers

We consider the singular perturbation problem,

\[ \epsilon u^* - uu' = 0 \quad x \in (-1,1), \]
\[ u(-1) = 0 \quad \text{and} \quad u(1) = -1. \]  \hspace{1cm} (1)

from Chang [1], O'Malley [2] and Miller [3]. Using the boundary layer detection theorem in Zhang [4], an exponential boundary layer of width proportional the singular perturbation parameter can be found to the left boundary. Because of the presence of the boundary layer, the numerical methods from Keller [5] will have to be adjusted accordingly. Choudhury [6] and Ilicasu [7] solved the problem using nonstandard high order methods on a uniform mesh. With the theory of weak formulation in Lax [8], Zhang [9] proposed test functions fitting the exponential boundary layer to solve the singular perturbation. In this paper, the singular perturbation problem is solved with improved high order methods based on the boundary layer detection theorem. Furthermore, a sixth order method is developed and verified numerically.

An improvement on the 4th order method

Let \( \omega = \frac{1}{\epsilon} \) where \( \epsilon \) is the singular perturbation parameter in the problem (1). Then we rewrite the second order derivative and compute the third and fourth order derivatives,

\[ u^* = \omega u', \]
\[ u'' = \omega (u')^2 + \omega u^*, \]

and...
\[ u^{(4)} = 2\omega u u'' + \omega u u'' + \omega u u' = 3\omega u u'' + \omega^2 u (u')^2 + \omega^2 u u' = \omega^2 u (u')^2 + \left(3\omega u' + \omega^2 u^2\right) u' \]

Setting \( A_3 = \omega u', B_3 = \omega u, \) and \( A_4 = \omega^2 u, B_4 = 3\omega u', \omega^2 u^2, \) we get \( u^{(3)} = A_3 u' + B_3 u'', \ u^{(4)} = A_4 u' + B_4 u'. \)

For the purpose of simplicity, we continue to use the following new notations,

\[ A^* = \frac{h^4 A_4}{24}, \ B^* = h^4 A_3, \ C^* = \frac{h^2}{2}, \ D^* = \frac{h^4 B_4}{6} \]

To improve the accuracy of the process, we develop the fourth order finite differences to approximate the singular perturbation problem (1).

\[ \epsilon u^{*} - u'u' \approx c_1 u_{i+1} + c_2^* u_i + c_3^* u_{i-1} \]

where

\[ c_1^* = \frac{-uD''D'' - \epsilon B'' + \epsilon A'' + uC}{2(A''D'' - B''C'')}, \]
\[ c_1^* = \frac{-uD''D'' - \epsilon B'' - \epsilon A'' + uC}{2(A''D'' - B''C'')}, \]
\[ c_1^* = \frac{-\left(c_1^* + c_1^*\right)}{2(A''D'' - B''C'')} \]

In Ilicasu [7], the derivative \( u_i' \) in \( A_3, A_4 \) and \( B_4 \) is replaced with \( u_i' = \frac{u_{i+1} - u_{i-1}}{2h} \). We now use the fourth order finite differences to establish a higher order approximation to the derivative \( u_i' \). For \( i = 1, 2, \ldots, N-1 \), let the first derivative be \( u_i' = c_1 u_{i+1} + c_2 u_i + c_3 u_{i-1} \) where \( c_1, c_2 \) and \( c_1 \) are constants.

By Taylor series expansion, we have

\[ u_i' = c_1 \left(u_i + hu' + \frac{h^2}{2} u'' + \frac{h^3}{6} u''' + \frac{h^4}{24} u^{(4)} + \ldots \right) \]
\[ + c_2 u_i + c_1 \left(u_i - hu' + \frac{h^2}{2} u'' - \frac{h^3}{6} u''' - \frac{h^4}{24} u^{(4)} + \ldots \right) \]
\[ = c_1 \left(u_i + hu' + \frac{h^2}{6} (\omega u'' + \omega u u') + \frac{h^3}{24} \left[\omega^2 u u' + \left(3\omega u' + \omega^2 u^2\right) u'\right] \right) \]
\[ + c_2 u_i + c_1 \left(u_i - hu' - \frac{h^2}{6} (\omega u'' + \omega u u') + \frac{h^3}{24} \left[\omega^2 u u' - \left(3\omega u' + \omega^2 u^2\right) u'\right] \right) \]
\[ = (c_1 + c_1) u_i + \left[(c_1 - c_1) \left(h + \frac{h^3}{6} \omega u'\right) + (c_1 + c_1) \frac{h^4}{24} \omega^2 u u'\right] u_i' \]
\[ + \left[(c_1 + c_1) \left(h^2 + \frac{h^4}{24} \omega u' + \frac{h^4}{24} \omega^2 u^2\right) + (c_1 - c_1) \frac{h^5}{6} \omega u\right] u_i' \]

Matching the corresponding coefficients, we obtain the following system of equations

\[ c_1 + c_2 + c_3 + 0, \]
\[ (c_1 - c_1) \left(h + \frac{h^3}{6} \omega u'\right) + (c_1 + c_1) \frac{h^4}{24} \omega^2 u u' = 1, \]
\[ (c_1 + c_1) \left(h^2 + \frac{h^4}{24} \omega u' + \frac{h^4}{24} \omega^2 u^2\right) + (c_1 - c_1) \frac{h^5}{6} \omega u = 0. \]

To understand the system better, we create more notations,
It is clear the system of equations is equivalent to
\[
\begin{align*}
c_3 + c_1 + c_1 &= 0, \\
(c_3 + c_1)A + (c_3 - c_1)B &= 1, \\
(c_3 + c_1)C + (c_3 - c_1)D &= 0,
\end{align*}
\]
of which, \( c_3 + c_1 \) and \( c_3 - c_1 \) are,

\[
\begin{align*}
c_3 + c_1 &= \frac{D}{AD - BC}, \\
(c_3 + c_1)A + (c_3 - c_1)B &= 1, \\
(c_3 + c_1)C + (c_3 - c_1)D &= 0,
\end{align*}
\]

Therefore, we get
\[
\begin{align*}
c_3 &= \frac{D - C}{2(AD - BC)}, \\
c_1 &= \frac{D + C}{2(AD - BC)}, \\
c_2 &= -(c_3 + c_1) = \frac{-D}{AD - BC}.
\end{align*}
\]

The error term is
\[
\eta = \frac{h}{120}(c_3 u^{(3)}(\eta) + c_1 u^{(5)}(\eta)) \quad \text{where} \quad \eta \in [x_i - h, x_i + h].
\]

Now \( c_3 \) and \( c_1 \) are updated to the fourth order accuracy. The improvement of the method is verified by numerical experiments.

**The 6th Order Method**

The second improvement is to add more terms from the Taylor series. We expand the \( u_{i+1} \) and \( u_{i-1} \) up to the sixth order derivatives. The following is a development of the sixth order method. The fifth order method is developed by dropping the sixth order derivative terms. We consider the fifth order and sixth order derivatives from the singular perturbation problem (1):

\[
u^{(3)} = 3\omega^2 u'' + 3\omega u''' + \omega^2 u''u' + 2\omega u' u'' + 2\omega^2 u' u'' + \omega^2 u''' + 2\omega u''' + \omega^2 u''
\]

\[
u^{(5)} = 3\omega^2 u'' + 3\omega u''' + \omega^2 u''u' + 2\omega u' u'' + 2\omega^2 u' u'' + \omega^2 u''' + 2\omega u''' + \omega^2 u''
\]

\[
u^{(7)} = 3\omega^2 u'' + 3\omega u''' + \omega^2 u''u' + 2\omega u' u'' + 2\omega^2 u' u'' + \omega^2 u''' + 2\omega u''' + \omega^2 u''
\]

\[
u^{(9)} = 4\omega^2 u'' + 4\omega u''' + 7\omega^2 u''u' + \omega^2 u''u' + \omega^2 u''u' + \omega^2 u''u'
\]

and
\( u^{(6)} = 6\omega u''u'' + 12\omega^2 u''u'' + 7\omega^3 \left[ (u''^2 + uu')u'' + uu'u'' \right] \\
+ 2\omega^3 u''u'' + 2\omega^3 u''u'' + 3\omega^3 u''u'' + \omega^3 uu'' \\
= 6\omega uu'' + 12\omega^2 uu'' + 7\omega^3 \left[ uu'' + uu'u'' \right] \\
+ 2\omega^3 uu'' + 5\omega^3 uu'' + \omega^3 uu'' \\
= 6\omega uu'' + 6\omega uu'' + 12\omega^2 uu'' + 7\omega^3 uu'' + 7\omega^3 uu'' + 7\omega^3 uu'' + 7\omega^3 uu'' \\
+ 2\omega u u + 5\omega^2 uu'' + \omega^4 uu'' + \omega^4 uu'' \\
= 25\omega uu'' + 13\omega uu'' + 7\omega uu'' + 7\omega uu'' + 12\omega uu'' + 7\omega uu'' + 7\omega uu'' \\
+ 2\omega uu'' + \omega^4 uu'' + \omega^4 uu'' \\
= \left( 9\omega uu'' + \omega^4 uu'' \right) u' + \left( 25\omega uu'' + 13\omega uu'' + 12\omega uu'' + \omega^4 u^4 \right) u^6.

For simplicity, we rewrite the derivatives

\( u^{(3)} = A_i u_i + B_i u^* \) where \( A_i = \omega u_i, B_i = \omega u^* \),

\( u^{(4)} = A_i u_i + B_i u^* \) where \( A_i = \omega^2 u_i, B_i = 3\omega u^* \),

\( u^{(5)} = A_i u_i + B_i u^* \) where \( A_i = 4\omega^2 u_i + \omega u^* u_i, B_i = 3\omega u^* + 7\omega^3 u_i + \omega^3 u^* \),

and

\( u^{(6)} = A_i u_i + B_i u^* \) where \( A_i = 9\omega^2 u_i u_i + \omega^3 u_i u_i, \) and

\( B_i = 25\omega u_i u_i + 13\omega^2 u_i u_i + 12\omega^3 u_i u_i + \omega^4 u_i \\
\)

We write

\( c_i u_i - u_i u_i = c_{i+1} u_i + c_i u_i + c_i u_i \\
\)

where \( c_i, c_2 \) and \( c_i \) are constants. By Taylor series expansion, we obtain

\( c_i u_i - u_i u_i = c_{i+1} u_i + c_i u_i + c_i u_i \\
\)

\( \approx c_3 \left[ u_i + h u_i' + h^2 u_i'' + \frac{h^3}{2} u_i''' + \frac{h^4}{6} u_i^{(4)} + \frac{h^5}{24} u_i^{(5)} + \frac{h^6}{120} u_i^{(6)} \right] \\
+ c_i u_i + c_i \left[ u_i - h u_i' + h^2 u_i'' - \frac{h^3}{6} u_i''' + \frac{h^4}{24} u_i^{(4)} - \frac{h^5}{120} u_i^{(5)} + \frac{h^6}{720} u_i^{(6)} \right] \\
= c_3 \left[ u_i + h u_i' + h^2 u_i'' + \frac{h^3}{6} (A_i u_i + B_i u_i) + \frac{h^4}{24} (A_i u_i + B_i u_i) + \frac{h^5}{120} (A_i u_i + B_i u_i) + \frac{h^6}{720} (A_i u_i + B_i u_i) \right] \\
+ c_i u_i + c_i \left[ u_i - h u_i' + h^2 u_i'' - \frac{h^3}{6} (A_i u_i + B_i u_i) + \frac{h^4}{24} (A_i u_i + B_i u_i) - \frac{h^5}{120} (A_i u_i + B_i u_i) + \frac{h^6}{720} (A_i u_i + B_i u_i) \right] \\
= c_3 \left[ u_i + \left( h + \frac{h^3 A_i}{6} + \frac{h^4 A_i}{120} + \frac{h^5 A_i}{720} \right) u_i' + \left( h^2 + \frac{h^3 B_i}{6} + \frac{h^4 B_i}{24} + \frac{h^5 B_i}{120} + \frac{h^6 B_i}{720} \right) u_i'' + \frac{h^6}{720} (A_i u_i + B_i u_i) \right] \\
+ c_i \left[ u_i - \left( h - \frac{h^3 A_i}{6} + \frac{h^4 A_i}{120} + \frac{h^5 A_i}{720} \right) u_i' + \left( h^2 + \frac{h^3 B_i}{6} + \frac{h^4 B_i}{24} + \frac{h^5 B_i}{120} + \frac{h^6 B_i}{720} \right) u_i'' \right] \\
= \left( c_3 + c_i + c_i \right) u_i + \left( c_3 + c_i \right) \left( \frac{h^3 A_i}{6} + \frac{h^4 A_i}{120} \right) u_i' + \left( c_3 + c_i \right) \left( \frac{h^5 A_i}{6} + \frac{h^6 A_i}{120} \right) u_i'' \\
+ \left( c_3 + c_i \right) \left( \frac{h^3 B_i}{6} + \frac{h^4 B_i}{24} + \frac{h^5 B_i}{120} \right) u_i' + \left( c_3 + c_i \right) \left( \frac{h^5 B_i}{6} + \frac{h^6 B_i}{120} \right) u_i''.

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Matching the corresponding coefficients of both sides, we get the following system of equation in terms of $c_i^*$, $c_i^*$ and $c_i^*$:

$$c_3^* + c_2^* + c_1^* = 0,$$

$$\left(c_3^* + c_i^*\right) A^* + \left(c_3^* - c_1^*\right) B^* = -u_i^*,$$

$$\left(c_3^* + c_i^*\right) C^* + \left(c_3^* - c_1^*\right) D^* = \varepsilon,$$

where

$$A^* = \frac{h^2 A_1}{24} + \frac{h^6 A_2}{720}, B^* = h + \frac{h^4 A_3}{6} + \frac{h^8 A_4}{120}, C^* = \frac{h^2}{2} + \frac{h^4 B_1}{24} + \frac{h^8 B_2}{720}, D^* = \frac{h^3 B_3}{6} + \frac{h^7 B_4}{120}.$$

Note that the derivatives contained in $A_i, A_i, A_i$ and $B_i, B_i, B_i$ are replaced with the following:

$$u_i^* = \frac{u_{i+1} - u_{i-1}}{2h}, u_i^* = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

Thus, we get

$$c_3^* + c_1^* = -\frac{u_i D^* - \varepsilon B^*}{A^* D^* - B^* C^*},$$

$$c_3^* - c_1^* = \frac{\varepsilon A^* + u_i C^*}{A^* D^* - B^* C^*}.$$

Therefore, the solution is

$$C_3^* = -\frac{u_i D^* - \varepsilon B^* - \varepsilon A^* + u_i C^*}{2\left(A^* D^* - B^* C^*\right)},$$

$$C_1^* = -\frac{-u_i D^* - \varepsilon B^* - \left(\varepsilon A^* + u_i C^*\right)}{2\left(A^* D^* - B^* C^*\right)},$$

$$c_2^* = -(c_3^* + c_1^*).$$

The error term is

$$\frac{h^2}{5040}\left(c_i u_i^{(7)}(\eta_1) + c_i u_i^{(7)}(\eta_4)\right)$$

where $\eta_3, \eta_4 \in [x_i - h, x_i + h].$

The results are improved as shown in the tables. We compare the numerical results among different methods. For the methods of Choudhury [6] and Ilicasu [7], a uniform mesh is used with N=2,000 mesh points. For the improved 4th order, 5th order and 6th order methods, the number $N_n$ of mesh points on the non-boundary layer domain is 170 and the number $N_b$ of points on the boundary layer is 300. In addition to the improved accuracy, the computing cost is reduced significantly thanks to fewer number of mesh points.

Table 1. Maximal error comparison among different methods $\varepsilon=0.01$

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of Points</th>
<th>Number of Iterations</th>
<th>Max Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choudhury’s Method [6]</td>
<td>2,000</td>
<td>Not known</td>
<td>2.91*10^-2</td>
</tr>
<tr>
<td>2nd order of Ilicasu [7]</td>
<td>2,000</td>
<td>3,201</td>
<td>2.61*10^-4</td>
</tr>
<tr>
<td>4th order of Ilicasu [7]</td>
<td>2,000</td>
<td>3,152</td>
<td>1.00*10^-5</td>
</tr>
<tr>
<td>Improved 4th order method of this paper</td>
<td>470</td>
<td>697</td>
<td>8.40*10^-5</td>
</tr>
<tr>
<td>5th order method of this paper</td>
<td>470</td>
<td>697</td>
<td>1.34*10^-7</td>
</tr>
<tr>
<td>6th order method of this paper</td>
<td>470</td>
<td>697</td>
<td>7.71*10^-8</td>
</tr>
</tbody>
</table>

For the methods of this paper, the tolerance of iteration is set at $10^{-10}$.

The improved high order methods work well for much smaller values of the singular perturbation parameter. The con-
vergence of the improved fourth order method is shown in Table 2, with the smallest values of singular perturbation parameter $\varepsilon = 10^{-12}$.

<table>
<thead>
<tr>
<th>Number of Points</th>
<th>$\varepsilon=10^{-5}$</th>
<th>$\varepsilon=10^{-10}$</th>
<th>$\varepsilon=10^{-12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N=350 \ (N_n=200, N_b=150)$</td>
<td>3.53*10^{-4}</td>
<td>3.53*10^{-4}</td>
<td>3.56*10^{-4}</td>
</tr>
<tr>
<td>$N=400 \ (N_n=200, N_b=200)$</td>
<td>1.91*10^{-4}</td>
<td>1.94*10^{-4}</td>
<td>1.94*10^{-4}</td>
</tr>
<tr>
<td>$N=450 \ (N_n=200, N_b=250)$</td>
<td>1.20*10^{-4}</td>
<td>1.22*10^{-4}</td>
<td>1.35*10^{-4}</td>
</tr>
<tr>
<td>$N=500 \ (N_n=200, N_b=300)$</td>
<td>8.39*10^{-5}</td>
<td>8.40*10^{-5}</td>
<td>9.71*10^{-5}</td>
</tr>
</tbody>
</table>

References


