Existence and Multiplicity of Solutions for Polyharmonic Elliptic Problem on Eistein Manifold Involving GJMS Operator

Kamel Tahri

High School of Management or Abou Bekr Belkaid University, Tlemcen (Algeria)

How to cite this paper: Tahri, K. (2018) Existence and Multiplicity of Solutions for Polyharmonic Elliptic Problem on Eistein Manifold Involving GJMS Operator. *Journal of Applied Mathematics and Computation*, 2(11), 513-524.
DOI: 10.26855/jamc.2018.11.003

*Corresponding author: Kamel Tahri, High School of Management or Abou Bekr Belkaid University, Tlemcen (Algeria).
Email: tahri_kamel@yahoo.fr

Abstract

Let (M^n, g) be a compact Eistein Riemannian manifold of dimension n and denote by S_g the scalar curvature of g . Let k be a positive integer. We prove that $n > 2k$ and under some geometric conditions that the elliptic equation:

$$P_g^k(u) = f(x) |u|^{2_k^*-2} u + \lambda h(x) |u|^{q-2} u$$

has two distinct solutions: one positive and other negative.

Keywords

1. Introduction.

In 1992, Graham-Jenne-Mason-Sparling in [8], discovered a particularly conformally fourth order operator defined on $2k$ -dimensional smooth Riemannian manifolds. This operator named in short GJMS operator. Given (M^n, g) a smooth, compact Riemannian manifold of dimension $n \geq 3$. Let k be a positive integer such that $2k < n$ and denote by S_g and Ric_g the scalar and Ricci curvature of g . Let $u \in C^{2k}(M)$, the GJMS has a principle part $(\Delta_g)^k u$, $k \in \mathbb{N}$. More precisely, for any riemannian metric g on M , the operator P_g^k is given by:

$$P_g^k : C^\infty(M) \rightarrow C^\infty(M)$$

such that for $u \in C^\infty(M)$

$$P_g^k(u) := (\Delta_g)^k u + lot$$

Where $\Delta_g u$ is the Laplace-Beltrami operator.

This operator is conformally invariant in the sense that if $\tilde{g} := \varphi^{\frac{4}{n-2k}} g$, for all $\varphi \in C^\infty(M)$, $\varphi > 0$:

$$P_g^k(\varphi u) = \varphi^{\frac{n+2k}{n-2k}} P_{\tilde{g}}^k(u).$$

Moreover, P_g^k is self-adjoint with respect to L^2 -scalar product. A scalar invariant is associated to this operator, namely the Q -curvature, denoted by Q_g^n . In the specific case $2k < n$, this geometric quantity has been obtained when we take $\varphi \equiv 1$ as

$$Q_g^k := \frac{2}{n-2k} P_g^k(1).$$

Observe that when (M^n, g) is Einstein, P_g^k is of constant coefficients, explicitly:

$$P_g^k(u) := \prod_{i=1}^{i=k} \left(\Delta_g - \frac{(n+2i-2) \cdot (n-2i) S_g}{4n(n-1)} \right) u$$

The GJMS operator enjoys an interesting conformal properties that are very similar to ones of conformal Laplacian and Paneitz-Branson operators. Recently, some remarkable developments have been achieved in the existence results of prescribed Q -curvature problem on Einsteinian manifolds, we refer the reader to [5], [6], [9] [11] and [12].

We denote by $H_k^2(M)$ is the standart Sobolev space, where for $1 \leq m \leq k$, $H_m^2(M)$ which the completion of $C^\infty(M)$ with respect to the norm:

$$\|u\|_{H_m^2(M)}^2 := \sum_{l=0}^{l=m} \int_M \left| \Delta_g^{\frac{l}{2}} u \right|^2 d\mu(g).$$

where

$$\Delta_g^{\frac{m}{2}} u := \begin{cases} \Delta_g^l u & \text{if } m = 2l \text{ is even} \\ \nabla_g \Delta_g^m u & \text{if } m = 2l + 1 \text{ is odd} \end{cases}$$

We denote by $K(n, k) > 0$ the best constant in the Sobolev's continuous embedding $D^{k,2}(\mathbb{R}^n) \hookrightarrow L_k^{2^*}(\mathbb{R}^n)$

$$K(n, k) := \inf_{u \in D^{k,2}(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} \left(\Delta^{\frac{k}{2}} u \right)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2_k^*} dx \right)^{\frac{2}{2_k^*}}} \quad (1)$$

Thanks to the series of results of Lions [17] and Swanson [19] that the extremal functions for the Sobolev inequality [18] exist and are exactly multiples of the functions:

$$u_\alpha(x) := \alpha_{n,k}^{\frac{n-2k}{4k}} \left(\frac{\lambda}{1 + \lambda^2 |x - a|^2} \right)^{\frac{n}{2_k^*}}$$

Where $a \in \mathbb{R}^n$, $\lambda > 0$,

$$\alpha_{n,k} = \prod_{l=-k}^{l=k-1} (n + 2l).$$

Now, we use the following Sobolev inequalities proven in to [18].

Theorem 1 Let (M^n, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$: For any $\epsilon > 0$, there is a constant $A_\epsilon \in \mathbb{R}$ such that

$$\forall u \in H_k^2(M) : \left(\int_M |u|^{2_k^*} d\mu(g) \right)^{\frac{2}{2_k^*}} \leq (1+\epsilon) K(n, k) \int_M \left| \Delta_g^{\frac{m}{2}} u \right|^2 d\mu(g) + A_\epsilon \|u\|_{H_{k-1}^2(M)}^2$$

Concerning the polyharmonic case, Pucci and Serrin in [8] have studied the following elliptic problem:

$$\begin{cases} (-\Delta)^k u = |u|^{2_k^*-2} u + \lambda u & \text{in } \Omega, \\ \left(\frac{\partial}{\partial \nu} \right)^j u \Big|_{\partial\Omega} = 0, j = 0, 1, 2, \dots, (k-1). \end{cases}$$

when Ω is the unit ball of $\mathbb{R}^n$ and they proved that it admits nontrivial radial symmetric solution for all $n \geq 8$ that if $\lambda \in (0, \lambda_1)$.

In [4], Benalili and Bekiri have established the existence of nodal solutions of the following polyharmonic elliptic problem on a Riemannian manifold with boundary:

$$\begin{cases} P_g^k(u) = f(x) |u|^{2_k^\#-2} u & \text{in } M, \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial M} = \varphi_j, \quad j = 0, 1, 2, \dots, (k-1). \end{cases}$$

where $\varphi_1, \dots, \varphi_{k-1}$ are smooth functions in ∂M .

Recently in [22], the author has been interested by the existence of multiple solutions of the Q -curvature problem with perturbation of the form:

$$P_g^k(u) = f(x) |u|^{2_k^\#-2} u + h \quad \text{in } M,$$

Where $f \in C^\infty(M)$ with $f > 0$ and h belongs to a suitable Sobolev space.

Recently in [24], the author has shown that the following nonhomogeneous Q -curvature problem with perturbation has at least two solutions of the form:

$$P_g^k(u) = f(x) |u|^{2_k^\#-2} u + h \quad \text{in } M,$$

Where $f \in C^\infty(M)$ with $f > 0$, $h \neq 0$ and $\|h\|_{(H_k^2(M))^*}$ is small enough and for every $k \geq 3$.

In this paper, We are concerned with the following problem. Let (M^n, g) be a smooth, compact Einstein Riemannian manifold of dimension $n \geq 3$ and denote by S_g and Ric_g the scalar and Ricci curvature of g . We investigate the existence and multiplicity of solutions for polyharmonic operators with critical exponent:

$$P_g^k(u) = f(x) |u|^{2_k^\#-2} u + \lambda h(x) |u|^{q-2} u.$$

Where $h \in L^d(M)$, with $d > \frac{2_k^\#}{2_k^\#-q} := \alpha$, $2 < q < N$ and $\lambda > 0$ a real parameter.

Throughout this section, we consider the energy functional J_λ , for each $u \in H_k^2(M)$,

$$J_\lambda(u) = \frac{1}{2} \int_M P_g^k(u) u \, d\mu(g) - \frac{\lambda}{q} \int_M h(x) |u|^q \, d\mu(g) - \frac{1}{2_k^\#} \int_M f(x) |u|^{2_k^\#} \, d\mu(g)$$

Here, we assume that:

(Hyp1) The operator $P_g^k : C^\infty(M) \longrightarrow C^\infty(M)$ is coercive i.e

$$\exists \Lambda > 0 : \langle P_g^k(u); u \rangle \geq \Lambda \|u\|_{H_k^2(M)}^2, \forall u \in H_k^2(M).$$

(Hyp2) The function $f : M \longrightarrow \mathbb{R}$ is positive C^∞ -function on M .

(Hyp3) If the function h doesn't vanish almost everywhere on M .

(Hyp4) The parameter $0 < \lambda < \lambda_1$ such that

$$\lambda_1 := \frac{q(2_k^\# - 2)}{2(2_k^\# - q)} \Lambda^{\frac{q}{2}} [\max((1 + \epsilon)K(n, k), A_\epsilon)]^{-\frac{q}{2}} \|h\|_\alpha^{-1}$$

2. Some Preparatory Lemmas

Before stating the proofs of our main results, we give some fruitful lemmas that help us in the following proofs:

Lemma 2 Assume that $P_g^k : C^\infty(M) \rightarrow C^\infty(M)$ is coercive. Then $|u|_{P_g^k} := \left[\int_M P_g^k(u) u d\mu(g) \right]^{\frac{1}{2}}$ is a norm on $H_k^2(M)$ equivalent to $\|u\|_{H_k^2(M)}$.

Proof. (See the proof in [19]).

Now, we first give an example of manifolds where the GJMS operator P_g^k is coercive and it will be our working manifold in the whole following analysis.

Lemma 3 Let (M^n, g) be a smooth Einstein Riemannian manifold of dimension $n \geq 3$ and denote by S_g its scalar curvature. Let k be a positive integer such that $2k < n$. Then, P_g^k is coercive if and only if $S_g > 0$.

Our working norm as follow, for all $u \in H_k^2(M)$:

$$|u|_{P_g^k} = \left[\int_M P_g^k(u) u d\mu(g) \right]^{\frac{1}{2}}$$

In this section the main tool is the Mountain-Pass lemma of Ambrosetti-Rabinowitz :

Lemma 4 let $J \in C^1(E, \mathbb{R})$ where $(E, \|\cdot\|_E)$ is a Banach space. We assume that:

(i) $J(0) = 0$.

(ii) $\exists r, R > 0$ such that $J(u) \geq R > 0$ for all $u \in E$ such that $\|u\|_E = r$.

(iii) $\exists v \in E$ such that $\limsup_{t \rightarrow +\infty} J(tv) < 0$.

Let

$$c = \min_{\eta \in \Gamma} \max_{t \in [0,1]} (J(\eta(t)))$$

where

$$\Gamma = \{ \eta \in C^1([0,1]; E) : \eta(0) = 0, \eta(1) = v \}$$

Then there exists a sequence $(u_n)_n$ in E such that :

$$J(u_n) \longrightarrow c \quad \text{and} \quad \nabla J(u_n) \longrightarrow 0 \quad \text{in } E^*$$

Moreover, we have that

$$c \leq \sup_{t \geq 0} J(tv)$$

We see that J_λ is a C^1 functional and its Fréchet derivative is given by:

$$\langle \nabla J_\lambda(u), v \rangle =$$

$$\int_M P_g^k(u) v d\mu(g) - \lambda \int_M h(x) |u|^{q-2} u v d\mu(g) - \int_M f(x) |u|^{2_k^*-2} u v d\mu(g).$$

We shall prove that the functional J_λ verifies the Mountain-Pass geometry conditions, namely.

Lemma 5 Suppose either or that. Then J_λ verifies the following conditions:

1. There exist constants $r, R > 0$, such that $J_\lambda(u) \geq R > 0$, $|u|_{P_g^k} = r$.
2. There exists $v \in H_k^2(M)$ with $|v|_{P_g^k} > r$, such that $J_\lambda(v) < 0$.

Proof.

1. We want to prove this lemma in two parts:

Let $u \in H_k^2(M)$ such that $|u|_{P_g^k} = r > 0$; then we have

$$J_\lambda(u) = \frac{1}{2} |u|_{P_g^k}^2 - \frac{\lambda}{q} \int_M h(x) |u|^q d\mu(g) - \frac{1}{2_k^\#} \int_M f(x) |u|^{2_k^\#} d\mu(g)$$

By Hölder and Sobolev inequalities, we deduce that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2} |u|_{P_g^k}^2 - \frac{\lambda}{q} (\max((1+\epsilon)K(n, k), A_\epsilon))^{\frac{q}{2}} \|h\|_\alpha \|u\|_{H_k^2(M)}^q \\ &\quad - \frac{1}{2_k^\#} (\max((1+\epsilon)K(n, k), A_\epsilon))^{\frac{2_k^\#}{2}} \max_{x \in M}(f(x)) \|u\|_{H_k^2(M)}^{2_k^\#} \end{aligned}$$

Hence if (Hyp1) is satisfied then, there exists a constant $\Lambda > 0$, such that :

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{2} |u|_{P_g^k}^2 - \frac{\lambda}{q} \Lambda^{-\frac{q}{2}} (\max((1+\epsilon)K_\circ, A_\epsilon))^{\frac{q}{2}} \|h\|_\alpha^q |u|_{P_g^k}^q \\ &\quad - \frac{1}{2_k^\#} \Lambda^{-\frac{N}{2}} (\max((1+\epsilon)K_\circ, A_\epsilon))^{\frac{2_k^\#}{2}} \max_{x \in M}(f(x)) |u|_{P_g^k}^{2_k^\#} \end{aligned}$$

With $|u|_{P_g^k} = r$, we get

$$\begin{aligned} J_\lambda(u) &\geq r^2 \times \left(\frac{1}{2} - \lambda \frac{\Lambda^{-\frac{q}{2}}}{q} (\max((1+\epsilon)K(n, k), A_\epsilon))^{\frac{q}{2}} \|h\|_\alpha r^{q-2} \right. \\ &\quad \left. - \frac{\Lambda^{-\frac{2_k^\#}{2}}}{2_k^\#} (\max((1+\epsilon)K(n, k), A_\epsilon))^{\frac{2_k^\#}{2}} \max_{x \in M}(f(x)) r^{2_k^\#-2} \right) \end{aligned}$$

Then there are $r, R > 0$, such that for all $u \in H_k^2(M)$ with $|u|_{P_g^k} = r$ we get $J_\lambda(u) \geq R > 0$.

2. Let $t > 0$ and $u \in H_k^2(M) - \{0\}$; thus

$$J_\lambda(tu) = \frac{t^2}{2} |u|_{P_g^k}^2 - \frac{\lambda t^q}{q} \int_M h(x) |u|^q d\mu(g) - \frac{t^{2_k^\#}}{2_k^\#} \int_M f(x) |u|^{2_k^\#} d\mu(g)$$

Since $2 < q < 2_k^\#$, we have that

$$J_\lambda(tu) \longrightarrow -\infty \text{ as } t \longrightarrow +\infty$$

This proves lemma.

Applying the Mountain-Pass lemma, there exists a sequence $(u_n)_n$ in $H_k^2(M)$ such that:

$$J(u_n) \longrightarrow c_\lambda \text{ and } \nabla J(u_n) \longrightarrow 0 \text{ in } (H_k^2(M))^*$$

Where

$$\Gamma = \{\eta \in C^1([0; 1]; H_k^2(M)) : \eta(0) = 0, \eta(1) = v\}$$

and

$$c_\lambda = \min_{\eta \in \Gamma} \max_{t \in [0, 1]} (J_\lambda(\eta(t)))$$

Let

$$\lambda_1 = \frac{q(2_k^\# - 2)}{2(2_k^\# - q)} \Lambda^{\frac{q}{2}} (\max((1 + \epsilon)K(n, k), A_\epsilon))^{-\frac{q}{2}} \|h\|_\alpha^{-1}$$

Then, we have the following lemmas.

Lemma 6 Let (M, g) is an n -dimensional compact, smooth and orientable Riemannian manifold with $n \geq 5$ with we suppose the conditions (h^1) , (h^2) and (h^3) are satisfying.

Then each Palais-Smale sequence at level c_λ is bounded in $H_k^2(M)$.

Proof.

Let $(u_m)_m$ a sequence in $H_k^2(M)$ such that :

$$J(u_m) \longrightarrow c_\lambda \quad \text{and} \quad \nabla J(u_m) \longrightarrow 0 \quad \text{in} \quad (H_k^2(M))^*$$

Then we have:

$$J(u_m) - \frac{1}{2_k^\#} \langle \nabla J(u_m), u_m \rangle = \left(\frac{1}{2} - \frac{1}{2_k^\#} \right) \|u_m\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{2_k^\#} \right) \int_M h(x) |u_m|^q dv(g)$$

By Hölder and Sobolev inequalities, we get that

$$\begin{aligned} J(u_m) - \frac{1}{2_k^\#} \langle \nabla J(u_m), u_m \rangle &= c_\lambda + o(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{2_k^\#} \right) \|u_m\|^2 - \left(\frac{1}{q} - \frac{1}{2_k^\#} \right) (\max((1 + \epsilon)K(n, k), A_\epsilon))^{\frac{q}{2}} \|h\|_\alpha \|u_m\|_{H_k^2(M)}^q \end{aligned}$$

By the coerciveness of the operator P_g^k , there exists a constant $\Lambda > 0$, such that :

$$c_\lambda + o(1) \geq \left(\frac{1}{2} - \frac{1}{2_k^\#} \right) \|u_m\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{2_k^\#} \right) \Lambda^{-\frac{q}{2}} (\max((1 + \epsilon)K(n, k), A_\epsilon))^{\frac{q}{2}} \|h\|_\alpha \|u_m\|^q$$

we consider the following when $\|u_m\| \geq 1$, then we have

$$c_\lambda + o(1) \geq \left(\frac{2_k^\# - 2}{2} - \lambda \frac{2_k^\# - q}{q} \Lambda^{-\frac{q}{2}} (\max((1 + \epsilon)K(n, k), A_\epsilon))^{\frac{q}{2}} \|h\|_\alpha \right) \frac{\|u_m\|^q}{2_k^\#}$$

Hence if (Hyp2) is satisfied then we have

$$\|u_m\| \leq \left(\frac{c_\lambda}{\frac{2_k^\# - 2}{22_k^\#} - \lambda \frac{2_k^\# - q}{q2_k^\#} \Lambda^{-\frac{q}{2}} (\max((1 + \epsilon)K(n, k), A_\epsilon))^{\frac{q}{2}} \|h\|_\alpha} \right)^{\frac{1}{q}} + o(1)$$

Thus, the sequence $(u_m)_m$ is bounded in $H_k^2(M)$.

Theorem 7 Let (M^n, g) is an n -dimensional compact, smooth and orientable Riemannian manifold with $n \geq 5$. Let be $(u_m)_m$ a Palais-Smale sequence at level c_λ and we suppose the conditions (Hyp1), (Hyp2) and (Hyp3) are satisfying.

Then there is a subsequence of $(u_m)_m$ converging strongly in $H_k^2(M)$.

Proof.

Let $(u_m)_m$ be a $(P.S)_{c_\lambda}$ sequence. By using Lemma and from the reflexivity of $H_k^2(M)$ and the compact imbedding

theorem, we conclude that there exists a subsequence $(u_m)_m$ and $u \in H_2^2(M)$ such that:

- (1). $u_m \rightharpoonup u$ weakly in $H_2^2(M)$.
- (2). $u_m \rightarrow u$ strongly in $L^p(M)$ for $1 < p < N = \frac{2n}{n-4}$.
- (3). $\nabla u_m \rightarrow \nabla u$ strongly in $L^q(M)$ for $1 < q < 2^* = \frac{2n}{n-2}$.
- (4). $u_m \rightarrow u$ a.e in M .

Then we deduce that

$$\left| \int_M a(x) |\nabla(u_m - u)|^2 dv(g) \right| \leq \|a\|_r \|\nabla(u_m - u)\|_{\frac{2r}{r-1}}^2 \rightarrow 0 \quad \text{strongly because } \frac{2r}{r-1} < 2^* = \frac{2n}{n-2}$$

and

$$\left| \int_M b(x) |u_m - u|^2 dv(g) \right| \leq \|b\|_s \|u_m - u\|_{\frac{2s}{s-1}}^2 \rightarrow 0 \quad \text{strongly since } \frac{2s}{s-1} < N = \frac{2n}{n-4}$$

and also

$$\left| \int_M h(x) |u_m - u|^q dv(g) \right| \leq \|h\|_d \|u_m - u\|_{\frac{dq}{d-1}}^q \rightarrow 0 \quad \text{strongly since } \frac{dq}{d-1} < N = \frac{2n}{n-4}$$

After these preliminaries we can prove that $v_m := u_m - u$ converges to 0 strongly in $H_2^2(M)$.

Using Brézis-Lieb Lemma, we obtain

$$\|\Delta_g u_m\|_2^2 - \|\Delta_g u\|_2^2 = \|\Delta_g v_m\|_2^2 + o(1)$$

and

$$\int_M f(x) (|u_m|^N - |u|^N) dv(g) = \int_M f(x) |v_m|^N dv(g) + o(1)$$

and also

$$\int_M h(x) (|u_m|^q - |u|^q) dv(g) = \int_M h(x) |v_m|^q dv(g) + o(1)$$

Taking account of $v_m \rightarrow 0$ weakly in $H_2^2(M)$, we have for all $\phi \in L^2(M)$

$$\int_M v_m \Delta_g^2 \phi dv(g) = o(1)$$

Then,

$$\int_M u \Delta_g^2 \phi dv(g) + o(1) = \int_M \Delta_g \phi \Delta_g u dv(g) + o(1)$$

Since $u_m \rightharpoonup u$ weakly in $H_2^2(M)$ i.e. for all $\phi \in (H_2^2(M))^*$, we have

$$\phi(v_m) = \phi(u_m - u^+) = o(1)$$

we take in particular $\phi = P_g^k(\varphi)$ for $\varphi \in H_2^2(M)$:

$$\int_M P_g(\varphi) v_m dv(g) = o(1)$$

If we take $\varphi = u$, we obtain

$$\int_M (\Delta_g u_m \Delta_g u - a(x)g \langle \nabla u_m, \nabla u \rangle + b(x)u_m u) dv(g) = \|u\|^2 + o(1)$$

and also if we put $\varphi = u_m$,

$$\int_M (\Delta_g u_m \Delta_g u - a(x)g \langle \nabla u_m, \nabla u \rangle + b(x)u_m u) dv(g) = \|u_m\|^2 + o(1)$$

and also

$$\int_M \left(\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u \right) dv(g) = \|u\|^2 + o(1)$$

And by letting $m \rightarrow +\infty$

$$\int_M \left(\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u \right) dv(g) \rightarrow \int_M \left(\lambda |u|^q + f(x) |u|^N \right) dv(g)$$

Taking account of we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2} \|\Delta_g(u_m - u)\|_2^2 - \frac{1}{N} \int_M f(x) |u_m - u|^N dv(g) + o(1)$$

And taking account of $u_m - u \rightarrow 0$ weakly in $H_2^2(M)$, we test by $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$

$$\begin{aligned} \langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u \rangle &= o(1) \\ &= \|\Delta_g(u_m - u)\|_2^2 - \int_M f(x) |u_m - u|^N dv(g) = o(1) \end{aligned}$$

So

$$\|\Delta_g(u_m - u)\|_2^2 = \int_M f(x) |u_m - u|^N dv(g) + o(1)$$

We take account of (??), we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2} \|\Delta_g(u_m - u)\|_2^2 - \frac{1}{N} \|\Delta_g(u_m - u)\|_2^2 + o(1)$$

i.e.

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{n} \|\Delta_g(u_m - u)\|_2^2$$

By Sobolev inequality, we have for all $u \in H_2^2(M)$

$$\|u\|_N^2 \leq (1 + \varepsilon) K_\circ \int_M (\Delta_g u)^2 + |\nabla_g u|^2 dv(g) + A_\varepsilon \int_M u^2 dv(g)$$

We test the Sobolev inequality by $u_m \rightharpoonup u$, we get

$$\|u_m - u\|_N^2 \leq (1 + \varepsilon) K_\circ \int_M (\Delta_g(u_m - u))^2 dv(g) + o(1)$$

So

$$\int_M f(x) |u_m - u|^N dv(g) \leq \max_{x \in M} f(x) \int_M |u_m - u|^N dv(g)$$

Hence taking account of (2), we obtain

$$\int_M f(x) |u_m - u|^N dv(g) \leq (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_{\circ}^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N + o(1)$$

Taking account of equality (2), one writes

$$o(1) \geq \|\Delta_g(u_m - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_{\circ}^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N + o(1)$$

Then

$$o(1) \geq \|\Delta_g(u_m - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_{\circ}^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N + o(1)$$

Taking account that $\|\cdot\|$ and $\|\cdot\|_{H_2^2(M)}$ are equivalent, there exists a constant $\Lambda > 0$ such that

$$o(1) \geq \|\Delta_g(u_m - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_{\circ}^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N + o(1)$$

$$o(1) \geq \|\Delta_g(u_m - u)\|_2^2 (1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_{\circ}^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^{N-2}) + o(1)$$

Hence if

$$\limsup_{m \rightarrow +\infty} \|\Delta_g(u_m - u)\|_2^{N-2} < \frac{1}{(1+\varepsilon)^{\frac{n}{n-4}} K_{\circ}^{\frac{n}{n-4}} \max_{x \in M} f(x)}$$

We get

$$\frac{2}{n} \int_M (\Delta_g(u_m - u))^2 dv(g) < c.$$

Since

$$c < \frac{2}{n K_{\circ}^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n-4}{4}}}$$

Then

$$\int_M (\Delta_g(u_m - u))^2 dv(g) < \frac{1}{K_{\circ}^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n-4}{4}}}.$$

Hence

$$o(1) \geq \underbrace{\|\Delta_g(u_m - u)\|_2^2 (1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_{\circ}^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^{N-2})}_{>0} + o(1)$$

Or also

$$\|\Delta_g(u_m - u)\|_2^2 = o(1)$$

i.e. $u_m \rightarrow u$ converges strongly in $H_2^2(M)$.

3. Positivity and Multiplicity of Solutions

We consider the following two functionals: for each $u \in H_k^2(M)$:

$$J_\lambda^+(u) = \frac{1}{2} \int_M P_g^k(u) u \, d\mu(g) - \frac{\lambda}{q} \int_M h(x) (u^+)^q \, d\mu(g) - \frac{1}{2_k^\#} \int_M f(x) (u^+)^{2_k^\#} \, d\mu(g)$$

And

$$J_\lambda^-(u) = \frac{1}{2} \int_M P_g^k(u) u \, d\mu(g) - \frac{\lambda}{q} \int_M h(x) (u^-)^q \, d\mu(g) - \frac{1}{2_k^\#} \int_M f(x) (u^-)^{2_k^\#} \, d\mu(g)$$

where

$$u^+ := \max(u, 0) \text{ and } u^- := \min(u, 0)$$

Then, critical points of J_λ^+ and J_λ^- are solutions to these equations respectively:

$$P_g^k(u) = f(x) u^{2_k^\#-1} + \lambda h(x) u^{q-1} \quad (2)$$

and

$$P_g^k(u) = \left(f(x) |u^-|^{2_k^\#-2} + \lambda h(x) |u^-|^{q-2} \right) u^-. \quad (3)$$

By applying the maximum principle k -times for the two previous equations and since $S_g > 0$; we obtain u^+ and u^- are solutions of the equations [13] and [14] respectively.

4. Test Functions

The purpose of this section is to prove that the geometric conditions obtained in [1] are verified. Let $(y^1, \dots, y^n)$ be normal coordinates centered at the point y_o where the function f attains its maximum. Let $B(y_o, \delta)$ be the ball centered at y_o and of radius δ with $0 < 2\delta < d$ (d the injectivity radius). Denote by $d\sigma$ the volume element of the $(n-1)$ -dimensional unit sphere S^{n-1} (1). We let also η be a smooth function equals to 1 on $B(y_o, \delta)$ and equals to 0 on $M - B(y_o, 2\delta)$.

Put

$$u_\epsilon(x) = \xi \cdot \left(\frac{\epsilon}{r^2 + \epsilon^2} \right)^{\frac{n-2k}{2}} \eta(r)$$

where

$$\xi := \left[\frac{\prod_{-k \leq j \leq k-1} (n+2j)}{f(a)} \right]^{\frac{n-2k}{4k}}$$

We know from the work of [20], that

$$\frac{1}{K(n, k)} = \prod_{-k \leq j \leq k-1} (n+2j) \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right)^{\frac{2k}{n}} \pi^k$$

where Γ denotes the Gamma function.

We compute now these quantities:

$$\zeta =: \int_M f(x) |u_\epsilon(x)|^{2_k^\sharp} d\mu(g)$$

and

$$\gamma := \int_M P_g^k(u_\epsilon) \cdot u_\epsilon d\mu(g)$$

As in [6], we get

$$\begin{aligned} \int_M f(x) |u_\epsilon(x)|^{2_k^\sharp} d\mu(g) &= \frac{1}{[K(n, k)]^{\frac{n}{2k}} [f(y_o)]^{\frac{n-2k}{2k}}} \times \\ &\left(1 - \frac{3\Delta f(y_o) + f(y_o)S_g(y_o)}{6(n-2)f(y_o)} \epsilon^2 + o(\epsilon^2) \right). \end{aligned}$$

and

$$\begin{aligned} \int_M P_g^k(u_\epsilon) \cdot u_\epsilon d\mu(g) &= \frac{1}{[K(n, k)]^{\frac{n}{2k}} [f(y_o)]^{\frac{n-2k}{2k}}} \times \\ &\left(1 - \left(\frac{(n^2 + 4n - 20)S_g(y_o) + 24(n-1)a(y_o)}{6(n^2 - 4)(n-6)} \right) \epsilon^2 + o(\epsilon^2) \right). \end{aligned}$$

We consider the path

$$\gamma_\epsilon(t) := \frac{u_\epsilon}{\|u_\epsilon\|_{2_k^\sharp}} t$$

It suffices to show that there exists $\epsilon > 0$ such that

$$\sup_{t \geq 0} J_\lambda(\gamma_\epsilon(t)) < \frac{k}{n [K(n, k)]^{\frac{n}{2k}} [f(y_o)]^{\frac{n-2k}{2k}}}$$

where

$$\begin{aligned} J_\lambda(\gamma_\epsilon(t)) &= \frac{t^2}{2 \|u_\epsilon\|_{2_k^\sharp}^2} \int_M P_g^k(u_\epsilon) \cdot u_\epsilon d\mu(g) - \\ &\frac{\lambda t^q}{q \|u_\epsilon\|_{2_k^\sharp}^q} \int_M h(x) |u_\epsilon|^q d\mu(g) - \frac{t^{2_k^\sharp}}{2_k^\sharp \|u_\epsilon\|_{2_k^\sharp}^{2_k^\sharp}} \int_M f(x) |u_\epsilon|^{2_k^\sharp} d\mu(g) \\ J_\lambda(\gamma_\epsilon(t)) &= \frac{t^2}{2} A_\epsilon - \frac{\lambda t^q}{q} B_\epsilon - \frac{t^{2_k^\sharp}}{2_k^\sharp} C_\epsilon \end{aligned}$$

Using the estimates above and we assume that

$$\frac{\Delta f(y_o)}{f(y_o)} < \frac{2(k+4)n^2 - 8(k+1)n - 24k}{3(n-2k)(n+2)(n-6)} S_g(y_o)$$

Then we get that

$$\sup_{t \geq 0} J_{\lambda}(\gamma_{\epsilon}(t)) < \frac{k}{n [K(n, k)]^{\frac{n}{2k}} [f(y_o)]^{\frac{n-2k}{2k}}}$$

Which achieves the proof completely.

Acknowledgement

The author is indebted to Prof. Mohammed Benalili for stimulating and helpful ideas and comments in all my career.

References

- [1] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* 14 (1973), 349-381.
- [2] T. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.* (9) 55 (1976), no. 3, 269-296.
- [3] M. Benalili, M. Békiri, Nodal solutions for elliptic equation involving the GJMS operators on compact manifolds (2017), arXiv:712.09069v1.
- [4] Z. Djadli, E. Hebey and M. Ledoux, Paneitz-type operators and applications, *Duke Math. J.* 104 (2000), no. 1, 129-169.
- [5] P. Esposito and F. Robert, Mountain pass critical points for Paneitz-Branson operators, *Calc. Var. Partial Differential Equations* 15 (2002), no. 4, 493-517.
- [6] C. Fefferman and Graham, *The Ambient metric*, Annals of Mathematics Studies, 178, Princeton University Press, Princeton, NJ, 2012. x+113 pp.
- [7] C. Graham, R. Jenne, L. Mason and G. Sparling, Conformally invariant powers of the Laplacian. I. Existence, *J. London Math. Soc.* (2) 46 (1992), no. 3, 557-565.
- [8] A. Gover, Laplacian operators and Q-curvature on conformally Einstein manifolds, *Math. Ann.* 336 (2006), no. 2, 311-334.
- [9] M. Gursky, F. Hang and Y. Lin, Riemannian manifolds with positive Yamabe invariant and Paneitz operator, preprint (2015), arXiv:1502.01050.32
- [10] M. Gursky and A. Malchiodi, A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature, to appear in *JEMS*, arXiv:1401.3216.
- [11] F. Hang and P. Yang, Q-curvature on a class of manifolds with dimension at least five, preprint (2014), arXiv:1411.3926.
- [12] A. Juhl, Explicit formulas for GJMS-operators and Q-curvatures, *Geom. Funct. Anal.* 23 (2013), no. 4, 1278-1370.
- [13] J. Lee and T. Parker, The Yamabe problem, *Bull. Amer. Math. Soc. (N.S.)* 17 (1987), no. 1, 37-91.
- [14] Y. Li and J. Xiong, Compactness of conformal metrics with constant Q-curvature. I, preprint (2015), arXiv:1506.00739.
- [15] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.* (2) 118 (1983), 349-374.
- [16] P. L. Lions, The concentration-compactness principle in the calculus of variations: The limit case. Part I, *Rev. Mat. Iberoamericana* 1 (1985), 145-201.
- [17] S. Mazumdar, GJMS-type Operators on a compact Riemannian manifold: Best constants and Coron-type solutions. *J. of Differential Equations* vol. 261 (2016) 4997-5034.
- [18] F. Robert, Admissible Q-curvatures under isometries for the conformal GJMS operators, *Nonlinear elliptic partial differential equations*, Contemp. Math., vol. 540, Amer. Math. Soc., Providence, RI, 2011, pp. 241-259.
- [19] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.
- [20] A. Swanson, The best Sobolev constant, *Appl. Anal.* 47 (1992), no. 4, 227-239.
- [21] K. Tahri, Multiple Solutions for the Nonhomogeneous of GJMS-operator, *EJMAA*, Jan 2018, 1-11.
- [22] K. Tahri, On GJMS-operator with Concave-convex Nonlinearities Involving Critical Sobolev Exponent (Preprint).
- [23] K. Tahri, Existence and Multiplicity Results of GJMS-operator (Preprint).
- [24] V. Wunsch, On conformally invariant differential operators, *Math. Nachr.* 129 (1986), 269-281.