On Some Transformations of ${}_2\psi_2$ Summation Formula And Their Applications

S. Ahmad Ali¹, Aditya Agnihotri¹

¹ Department of Mathematics, Babu Banarasi Das University, Lucknow 226028 (India).

How to cite this paper: Ahmad Ali, S., Agnihotri, A. (2018) On Some Transformations of ${}_2\psi_2$ Summation Formula And Their Applications. *Journal of Applied Mathematics and Computation*, 2(10), 456-465.

DOI: 10.26855/jamc.2018.10.001

*Corresponding author: S. Ahmad Ali, Department of Mathematics, Babu Banarasi Das University, Lucknow 226028 (India). Email: ali.sahmad@yahoo.com

Abstract

In the present paper, we have established some new transformations of a known summation formula for basic bilateral hypergeometric series. These transformations are capable of further generalizations and contain many known as well as new identities as their special cases including those of q -gamma, q -beta and eta-functions. Some of the applications have been mentioned.

Keywords

Basic bilateral hypergeometric series, Ramanujan's ${}_1\psi_1$ Sum, q -series, eta-function, q -gamma function.

2010 Mathematics Subject Classification. 33D15, 11F20, 33D05.

1. Introduction.

One of the most celebrated identity of Ramanujan is his ${}_1\psi_1$ sum recorded in his Notebook [1]. In modern notation this formula is read as

$${}_1\psi_1(a; b; q, z) = \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(q, b/a, az, q/az)_{\infty}}{(b, q/a, z, b/az)_{\infty}}, \quad (1.1)$$

where $|b/a| < |z| < 1$. Obviously, (1.1) is a bilateral generalization of q -binomial and was first discovered by Ramanujan. It was brought to the attention of mathematical world by Hardy [2] who described it as “a remarkable formula with many parameters”. The Ramanujan's ${}_1\psi_1$ identity has been extensively studied and have got numerous applications in many disciplines. For details one may refer to an elegant survey by Warnaar [3].

Recently, Adiga et al. [4], Anitha [5] and Ali et al. [6] have used (1.1) to develop some new transformation formulae and have produced a number of q -gamma and eta-function identities as an application of the new transformations of (1.1). In continuation of the works of Adiga et al. [4], Anitha [5] and Ali et al. [6], in the present work we have obtained the transformations of the following ${}_2\psi_2$ sum [7]

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n (bc/azq)_n}{(b)_n (c)_n} z^n = \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty} (c/a)_{\infty} (bc/azq)_{\infty}}{(z)_{\infty} (b)_{\infty} (c)_{\infty} (b/az)_{\infty} (c/az)_{\infty} (q/a)_{\infty}}. \quad (1.2)$$

Throughout this paper, we have used the following notations and definitions [8]. The generalized basic hypergeometric series is defined by

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, & a_2 & \dots & a_{r+1}; & q; z \\ b_1 & b_2 & \dots & b_r \end{matrix} \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_{r+1})_n}{(q)_n (b_1)_n \dots (b_r)_n} z^n,$$

where $|z| < 1$, $|q| < 1$ and a bilateral basic hypergeometric series is defined by

$${}_r\psi_r \left(\begin{matrix} a_1, & a_2 & \dots & a_r; & q; z \\ b_1 & b_2 & \dots & b_r \end{matrix} \right) = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(b_1)_n \dots (b_r)_n} z^n,$$

where $\left| \frac{b_1 \dots b_r}{a_1 \dots a_r} \right| < |z| < 1$, $|q| < 1$. Here $(a; q^k)_n$ is q -shifted factorial and

$$(a; q^k)_n = (1-a)(1-aq^k)(1-aq^{2k}) \dots (1-aq^{k(n-1)})$$

For $k=1$, we write

$$(a; q)_n = (a)_n,$$

Also

$$(a)_\infty = (a; q)_\infty = \prod_{n=0}^{\infty} (1-aq^n).$$

where $|q| < 1$.

We also recall that the q -difference operator D_q and the q -shifted operator ζ are defined as

$$D_q f(a) = \frac{f(a) - f(aq)}{a}$$

and

$$\zeta(f(a)) = f(aq).$$

Chen and Liu [9,10] have constructed an operator θ as

$$\theta = \zeta^{-1} D_q,$$

and defined the following operator $E(b\theta)$

$$E(b\theta) = \sum_{k=0}^{\infty} \frac{(b\theta)^k q^{k(k-1)/2}}{(q)_k}.$$

The following identities hold for $E(b\theta)$ [11, Theorem 1].

$$E(b\theta)(at; q)_\infty = (at, bt; q)_\infty, \quad (1.3)$$

$$E(b\theta)(as, at; q)_\infty = \frac{(as, at, bs, bt; q)_\infty}{(abst/q; q)_\infty} \quad (1.4)$$

where $|abst/q| < 1$.

Somashekara et al. [7] have obtained (1.2) by the method of parameter augmentation by using the identities (1.3) and (1.4) of the operator $E(b\theta)$. The operator $E(b\theta)$ along with (1.3) and (1.4) have also been used by Zhang and Wang [12] to discover many new q -series identities.

In the next section, we shall also require the following transformations [6].

If $|q/z| < |z| < 1$ and $|q| < 1$, then

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = -1 + \frac{(q, az)_{\infty}}{(b, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n (z)_n}{(q)_n (az)_n} q^n$$

$$+ \frac{(q/z)_{\infty}}{(b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(1/a)_n (b/a)_n}{(q)_n (q/a)_n} (q/z)^n,$$
(1.5)

If $|q/b| < |z| < 1$ and $|qaz/b| < 1$, then

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = -1 + \frac{(qaz/b)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n (b/a)_n}{(q)_n (b)_n} (qaz/b)^n$$

$$+ \frac{(q/b, qb/az)_{\infty}}{(q/a, b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (b/az)_n}{(q)_n (bq/az)_n} (q/b)^n,$$
(1.6)

If $|q| < |z| < 1$, then

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = -1 + \frac{(q, az)_{\infty}}{(b, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n (z)_n}{(q)_n (az)_n} q^n$$

$$+ \frac{(q/z)_{\infty}}{(1/z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(qz)_n (q/a)_n} q^n + \frac{(q, q/az, b/a)_{\infty}}{(q/a, b/az, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/az)_n}{(q)_n (q/az)_n} q^n,$$
(1.7)

If $|q/b| < |z| < 1$ and $|q| < 1$, then

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = -1 + \frac{(q, az)_{\infty}}{(b, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n (z)_n}{(q)_n (az)_n} q^n$$

$$+ \frac{(q/b, qb/az)_{\infty}}{(q/a, b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (b/az)_n}{(q)_n (qb/az)_n} (q/b)^n,$$
(1.8)

If $|q| < |z| < 1$, then

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = -1 + \frac{(qaz/b)_{\infty}}{(az/b)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(b)_n (bq/az)_n} q^n$$

$$+ \frac{(q, az, b/a)_{\infty}}{(b, az, b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n}{(q)_n (az)_n} q^n + \frac{(q/z)_{\infty}}{(1/z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(qz)_n (q/a)_n} q^n$$

$$+ \frac{(q, q/az, b/a)_{\infty}}{(q/a, b/az, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/az)_n}{(q)_n (q/az)_n} q^n.$$
(1.9)

If $|b/a| < |z| < 1$ and $|q| < 1$, then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(b/a, az)_{\infty}}{(b, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(qza/b)_n (a)_n}{(q)_n (az)_n} (b/a)^n \\ &+ \frac{(q/z)_{\infty}}{(1/z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(qz)_n (q/a)_n} q^n + \frac{(q, q/az, b/a)_{\infty}}{(q/a, b/az, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/az)_n}{(q)_n (q/az)_n} q^n, \end{aligned} \quad (1.10)$$

If $|b/a| < |z| < 1$ and $|q| < 1$, then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(qaz/b)_{\infty}}{(az/b)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(b)_n (bq/az)_n} q^n \\ &+ \frac{(b/a, q/az)_{\infty}}{(q/a, b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(q/z)_n (q/b)_n}{(q)_n (q/az)_n} (b/a)^n \\ &+ \frac{(q, az, b/a)_{\infty}}{(b, z, b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n}{(q)_n (az)_n} q^n, \end{aligned} \quad (1.11)$$

If $|b/a| < |z| < 1$ and $|q| < |z| < 1$, then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(b/a, az)_{\infty}}{(b, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(qza/b)_n (a)_n}{(q)_n (az)_n} (b/a)^n \\ &+ \frac{(q/z)_{\infty}}{(b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(1/a)_n (b/a)_n}{(q)_n (q/a)_n} (q/z)^n, \end{aligned} \quad (1.12)$$

If $|b/a| < |z| < 1$ and $|qaz/b| < |z| < 1$, then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(qaz/b)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n (b/a)_n}{(q)_n (b)_n} (qaz/b)^n \\ &+ \frac{(b/a, q/az)_{\infty}}{(q/a, b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(q/z)_n (q/b)_n}{(q)_n (q/az)_n} (b/a)^n, \end{aligned} \quad (1.13)$$

If $|q/z| < |z| < 1$ and $|qaz/b| < |z| < 1$, then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n &= -1 + \frac{(qaz/b)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n (b/a)_n}{(q)_n (b)_n} (qaz/b)^n \\ &+ \frac{(q/z)_{\infty}}{(b/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(1/a)_n (b/a)_n}{(q)_n (q/a)_n} (q/z)^n \end{aligned} \quad (1.14)$$

We shall also need the following transformations of ${}_3\psi_2$ series [8]

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(q)_n (d)_n (e)_n} (de/abc)^n \\ = \frac{(e/a, de/bc)_{\infty}}{(e, de/abc)_{\infty}} \sum_{n=0}^{\infty} \frac{(a)_n (d/b)_n (d/c)_n}{(q)_n (d)_n (de/bc)_n} (e/a)^n, \end{aligned} \quad (1.15)$$

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(q)_n(d)_n(e)_n} (de/abc)^n = \frac{(b, de/ab, de/bc)_{\infty}}{(d, e, de/abc)_{\infty}} \sum_{n=0}^{\infty} \frac{(e/b)_n(d/b)_n(de/abc)_n}{(q)_n(de/ab)_n(de/bc)_n} b^n. \quad (1.16)$$

2. Main Results

In this section, we establish the following transformations of ${}_2\psi_2$ which are true whenever the series involved have a meaning.

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n = -1 + \frac{(bc/azq, c/q, q, az)_{\infty}}{(b, c, z, bc/azq^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n(b/q)_n}{(q)_n(az)_n} q^n + \frac{(q/z, c/a, bc/azq)_{\infty}}{(bc/aq, b/az, c/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(1/a)_n(b/a)_n}{(q)_n(q/a)_n} (q/z)^n. \quad (2.1)$$

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n = -1 + \frac{(aqz/b)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n(b/a)_n}{(q)_n(b)_n} (aqz/b)^n + \frac{(q/b, qb/az, bc/azq, c/a)_{\infty}}{(q/a, b/az, c/az, bc/aq)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(b/az)_n}{(q)_n(qb/az)_n} (q/b)^n. \quad (2.2)$$

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n = -1 + \frac{(q, az, bc/azq, c/q)_{\infty}}{(z, b, c, bc/azq^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n(z)_n}{(q)_n(az)_n} q^n + \frac{(q/z)_{\infty}}{(1/z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(q/a)_n(qz)_n} q^n + \frac{(q, q/az, b/a, c/a, bc/aqz)_{\infty}}{(q/a, b/az, z, bc/aq, c/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/az)_n}{(q)_n(q/az)_n} q^n. \quad (2.3)$$

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n = -1 + \frac{(q, az, bc/azq, c/q)_{\infty}}{(z, b, c, bc/azq^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n(z)_n}{(q)_n(az)_n} q^n + \frac{(q/b, qb/az, bc/azq, c/a)_{\infty}}{(q/a, b/az, c/az, bc/aq)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(b/az)_n}{(q)_n(qb/az)_n} (q/b)^n. \quad (2.4)$$

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n = -1 + \frac{(qaz/b)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n(b/a)_n}{(q)_n(b)_n} (qaz/b)^n + \frac{(q/z, bc/azq, c/a)_{\infty}}{(b/az, c/az, bc/aq)_{\infty}} \sum_{n=0}^{\infty} \frac{(1/a)_n(b/a)_n}{(q)_n(q/a)_n} (q/z)^n. \quad (2.5)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n &= -1 + \frac{(az, b/a, c/a, bc/azq)_{\infty}}{(z, b, c, bc/a^2zq)_{\infty}} \sum_{n=0}^{\infty} \frac{(a)_n(aqz/b)_n}{(q)_n(az)_n} (b/a)^n \\ &\quad + \frac{(q/z)_{\infty}}{(1/z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(qz)_n(q/a)_n} q^n \\ &\quad + \frac{(q, q/az, b/a, c/a, bc/aqz)_{\infty}}{(q/a, b/az, z, bc/aq, c/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/az)_n}{(q)_n(q/az)_n} q^n. \end{aligned} \quad (2.6)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n &= -1 + \frac{(aqz/b)_{\infty}}{(az/b)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n}{(b)_n(bq/az)_n} q^n \\ &\quad + \frac{(q/az, bc/azq, b/a, c/a)_{\infty}}{(q/a, bc/aq, b/az, c/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(q/z)_n(q/b)_n}{(q)_n(q/az)_n} (b/a)^n \\ &\quad + \frac{(q, az, bc/azq, b/a, c/a)_{\infty}}{(b, c, b/az, c/az, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n}{(q)_n(az)_n} q^n. \end{aligned} \quad (2.7)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n &= -1 + \frac{(az, bc/azq, b/a, c/a)_{\infty}}{(z, b, c, bc/a^2zq)_{\infty}} \sum_{n=0}^{\infty} \frac{(a)_n(aqz/b)_n}{(q)_n(az)_n} (b/a)^n \\ &\quad + \frac{(q/az, bc/azq, c/a)_{\infty}}{(b/az, bc/aq, c/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(1/a)_n(b/a)_n}{(q)_n(q/a)_n} (q/z)^n. \end{aligned} \quad (2.8)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n &= -1 + \frac{(aqz/b)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/q)_n(b/a)_n}{(q)_n(b)_n} (aqz/b)^n \\ &\quad + \frac{(q/az, bc/azq, b/a, c/a)_{\infty}}{(q/a, bc/aq, b/az, c/az)_{\infty}} \sum_{n=0}^{\infty} \frac{(q/z)_n(q/b)_n}{(q)_n(q/az)_n} (b/a)^n. \end{aligned} \quad (2.9)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n &= -1 + \frac{(c/q, qz)_{\infty}}{(c, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(aqz/c)_n}{(b)_n(zq)_n} (c/q)^n \\ &\quad + \frac{(q/b, qz/c, qz)_{\infty}}{(q/a, q^2az/bc, q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(aqz/c)_n}{(qz/c)_n(qz)_n} (q/b)^n. \end{aligned} \quad (2.10)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n &= -1 + \frac{(a, bc/aq, qz)_{\infty}}{(b, c, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(c/a)_n(z)_n}{(q)_n(bc/aq)_n(zq)_n} a^n \\ &\quad + \frac{(qaz/bc, qz)_{\infty}}{(q^2az/bc, q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(c/a)_n}{(q/a)_n(qz)_n} (aqz/bc)^n. \end{aligned} \quad (2.11)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a)_n(bc/azq)_n}{(b)_n(c)_n} z^n &= -1 + \frac{(c/q, qz)_{\infty}}{(c, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(aqz/c)_n}{(b)_n(zq)_n} (c/q)^n \\ &\quad + \frac{(qaz/bc, qz)_{\infty}}{(q^2az/bc, q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n(c/a)_n}{(q/a)_n(qz)_n} (aqz/bc)^n. \end{aligned} \quad (2.12)$$

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n (bc/azq)_n}{(b)_n (c)_n} z^n = -1 + \frac{(a, bc/azq, qz)_{\infty}}{(b, c, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (c/a)_n (z)_n}{(q)_n (bc/azq)_n (zq)_n} a^n$$

$$+ \frac{(q/b, qz/c, qz)_{\infty}}{(q/a, q^2 az/bc, q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (aqz/c)_n}{(qz/c)_n (qz)_n} (q/b)^n. \quad (2.13)$$

Proof of (2.1): Employing Ramanujan's ${}_1\psi_1$ summation formula, (1.5) can be written as

$$\frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty}}{(z)_{\infty} (q/a)_{\infty}}$$

$$= -\{(b)_{\infty} (b/az)_{\infty}\} + \frac{(q)_{\infty} (az)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^n}{(q)_n (az)_n (bq^{n-1})_{\infty}} \{(b/az)_{\infty} (b/q)_{\infty}\}$$

$$+ (q/z)_{\infty} \sum_{n=0}^{\infty} \frac{(1/a)_n (q/z)^n}{(q)_n (q/a)_n (bq^n/a)_{\infty}} \{(b)_{\infty} (b/a)_{\infty}\}. \quad (2.14)$$

Applying $E(c\theta)$ on both sides with respect to b , we obtain

$$\frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty} (c/a)_{\infty}}{(z)_{\infty} (q/a)_{\infty}} = -\frac{(b)_{\infty} (b/az)_{\infty} (c)_{\infty} (c/az)_{\infty}}{(bc/azq)_{\infty}}$$

$$+ \frac{(q)_{\infty} (az)_{\infty}}{(z)_{\infty}} \sum_{n=0}^{\infty} \frac{(z)_n q^n}{(q)_n (az)_n (bq^{n-1})_{\infty}} \left\{ \frac{(b/az)_{\infty} (b/q)_{\infty} (c/az)_{\infty} (c/q)_{\infty}}{(bc/azq^2)_{\infty}} \right\}$$

$$+ (q/z)_{\infty} \sum_{n=0}^{\infty} \frac{(1/a)_n (q/z)^n}{(q)_n (q/a)_n (bq^n/a)_{\infty}} \left\{ \frac{(b)_{\infty} (b/a)_{\infty} (c)_{\infty} (c/a)_{\infty}}{(bc/aq)_{\infty}} \right\}.$$

Multiplying throughout by $\frac{(bc/azq)_{\infty}}{(b, b/az, c, c/az)_{\infty}}$ and using (1.2), we obtain (2.1).

Proofs of (2.2)-(2.9): Employing Ramanujan's ${}_1\psi_1$ summation formula in each of the identities (1.6)-(1.8), (1.14) and (1.10)-(1.13) and writing them in the form (2.14). Next, by applying the operator $E(c\theta)$ on both sides followed by multi-

plication by $\frac{(bc/azq)_{\infty}}{(b, b/az, c, c/az)_{\infty}}$, we obtain (2.2)-(2.9) respectively.

Proof of (2.10): We have

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n (bc/azq)_n}{(b)_n (c)_n} z^n$$

$$= -1 + \sum_{n=0}^{\infty} \frac{(a)_n (bc/azq)_n}{(b)_n (c)_n} z^n + \sum_{n=0}^{\infty} \frac{(q/c)_n (q/b)_n}{(q/a)_n (q^2 az/bc)_n} q^n. \quad (2.15)$$

Taking $a = q$, $b = a$, $c = bc/azq$, $d = b$ and $e = c$ in (1.15), we obtain

$$\sum_{n=0}^{\infty} \frac{(a)_n (bc/azq)_n}{(b)_n (c)_n} z^n = \frac{(c/q, qz)_{\infty}}{(c, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (azq/c)_n}{(b)_n (qz)_n} (c/q)^n. \quad (2.16)$$

Changing a to q , b to q/b , c to q/c , de/abc to q , d to q/a and e to $q^2 az/bc$ in (1.16), we obtain

$$\sum_{n=0}^{\infty} \frac{(q/b)_n (q/c)_n}{(q/a)_n (q^2 az/bc)_n} q^n = \frac{(q/b, qz/c, qz)_{\infty}}{(q/a, q^2 az/bc, q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (azq/c)_n}{(qz)_n (qz/c)_n} (q/b)^n. \quad (2.17)$$

Substituting the values from (2.16) and (2.17) in (2.15), we get (2.10).

Proof of (2.11): Putting $a = q$, $b = q/b$, $c = q/c$, $d = q/a$ and $e = q^2 az/bc$ in (1.15), we obtain

$$\sum_{n=0}^{\infty} \frac{(q/b)_n (q/c)_n}{(q/a)_n (q^2 az/bc)_n} q^n = \frac{(aqz/bc, qz)_{\infty}}{(q^2 az/bc, q)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (c/a)_n}{(q/a)_n (qz)_n} (aqz/bc)^n. \quad (2.18)$$

Taking $a = q$, $b = a$, $c = bc/azq$, $d = b$ and $e = c$ in (1.16), we obtain

$$\sum_{n=0}^{\infty} \frac{(a)_n (bc/azq)_n}{(b)_n (c)_n} z^n = \frac{(a, bc/aq, qz)_{\infty}}{(b, c, z)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/a)_n (c/a)_n (z)_n}{(q)_n (bc/aq)_n (qz)_n} a^n. \quad (2.19)$$

Substituting the values from (2.18) and (2.19) in (2.15), we get (2.11).

Proof of (2.12): Putting the values from (2.16) and (2.18) in (2.15), we obtain (2.12).

Proof of (2.13): Putting the values from (2.17) and (2.19) in (2.15), we obtain (2.13).

3. Some Applications

The results (2.1)-(2.13) can be used to reproduce a number of known summations, transformations and q -series identities as well as further new identities of unilateral and bilateral basic hypergeometric series. In this section, we mention only some of the consequences of the results of previous section.

In (2.6), if we employ ${}_2\psi_2$ sum (1.2) in L.H.S., and then putting $a = q^{1/2}$; $b = q$; $z = q^{1/2}$; $c = q^{1/2}$, we obtain

$$\sum_{n=0}^{\infty} \frac{q^n}{(q^{3/2})_n} = \frac{(1/q^{1/2})_{\infty}}{(q^{1/2})_{\infty}}.$$

Change of base q to q^2 and after simplification gives

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q^3; q^2)_n} = \frac{(q-1)}{q}. \quad (3.1)$$

In (2.5), if we employ ${}_2\psi_2$ sum (1.2) in L.H.S., and then putting $a = q$; $b = q^2$; $z = q^{1/2}$; $c = q$ and after changing the base q to q^2 , we obtain

$$\sum_{n=0}^{\infty} \frac{(q^2; q^2)_n q^n}{(q^4; q^2)_n} = 1. \quad (3.2)$$

If we take $c = q$ in (2.4) and then applying q Gauss summation formula in L.H.S., we obtain

$$\sum_{n=0}^{\infty} \frac{(b/a)_n (b/az)_n}{(q)_n (bq/az)_n} (q/b)^n = \frac{(b/a)_{\infty} (az)_{\infty} (q/az)_{\infty} (b/a)_{\infty}}{(b)_{\infty} (z)_{\infty} (q/b)_{\infty} (bq/az)_{\infty}} + \frac{(q/az)_{\infty} (b/a)_{\infty}}{(q/b)_{\infty} (bq/az)_{\infty}}.$$

On taking $z = 1/a$, we get the following summation formula

$$\sum_{n=0}^{\infty} \frac{(b/a)_n (b)_n}{(q)_n (bq)_n} (q/b)^n = \frac{(q, b/a)_{\infty}}{(bq, q/b)_{\infty}}. \quad (3.3)$$

Applying ${}_2\psi_2$ sum in (2.1) and then putting $a = 1/q$, $b = q^{1/2}$; $z = q^{1/2}$; $c = q^{1/2}$, we obtain

$$\frac{(q^{-1/2})_\infty (q^{3/2})_\infty (q^{3/2})_\infty (q^{3/2})_\infty}{(q^2)_\infty (q)_\infty (q^{1/2})_\infty (q^{1/2})_\infty} = -1 + \frac{(q)_\infty (q^{-1/2})_\infty}{(q^{1/2})_\infty (q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(q^{1/2})_n}{(q)_n} q^n \\ + \frac{(q^{1/2})_\infty (q^{3/2})_\infty (q^{1/2})_\infty}{(q)_\infty (q)_\infty (q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{3/2})_n}{(q^2)_n} q^{n/2}.$$

Changing the base q to q^2 yields

$$\frac{\eta^2(\tau)}{\eta^4(2\tau)} = \frac{(1-q)q^{3/4}}{1+q} + \frac{q^{-1/4}(1-q)^2(q^2; q^2)_\infty}{(1+q)(q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q; q^2)_n}{(q^2; q^2)_n} q^n \\ - \frac{q^{3/4}(1-q)}{1+q} \frac{(q; q^2)_\infty (q^3; q^2)_\infty^2}{(q^2; q^2)_\infty^3} \sum_{n=0}^{\infty} \frac{(q^3; q^2)_n}{(q^4; q^2)_n} q^n. \quad (3.4)$$

where

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2n\pi i \tau}) = q^{1/24} (q; q)_\infty$$

is the Dedekind eta-function, where $q = e^{2\pi i \tau}$ and $\text{Im} \tau > 0$.

Applying ${}_2\psi_2$ sum in (2.2) and then putting $a = 1/q$, $b = q^{1/2}$; $z = q^{1/2}$; $c = q^{3/2}$, we obtain

$$\frac{(q^{-1/2})_\infty (q^{3/2})_\infty (q^{5/2})_\infty (q^{3/2})_\infty}{(q^{1/2})_\infty (q^2)_\infty (q^2)_\infty (q^{1/2})_\infty} = -1 + \frac{(q^{1/2})_\infty (q^{3/2})_\infty (q^{5/2})_\infty}{(q)_\infty (q^2)_\infty (q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^{3/2})_n}{(q^2)_n} q^{n/2}.$$

Changing the base q to q^2 yields

$$\frac{\eta(\tau)}{\eta^2(2\tau)} = \frac{q^{7/8}(q^4; q^2)_\infty}{(1+q)(q^5; q^2)_\infty} - \frac{q^{7/8}}{1+q} \frac{(q; q^2)_\infty (q^3; q^2)_\infty}{(q^2; q^2)_\infty (q^4; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^3; q^2)_n}{(q^4; q^2)_n} q^n. \quad (3.5)$$

Changing a to q^a , b to q^b , c to q^c and z to q^z in (2.13), we obtain

$$\frac{\Gamma_q(z)\Gamma_q(b)\Gamma_q(c)\Gamma_q(b-a-z)\Gamma_q(c-a-z)\Gamma_q(1-a)}{\Gamma_q(a+z)\Gamma_q(b-a)\Gamma_q(1-a-z)\Gamma_q(c-a)\Gamma_q(b+c-a-z-1)} \\ = -1 + \frac{\Gamma_q(b)\Gamma_q(c)\Gamma_q(z)}{\Gamma_q(a)\Gamma_q(b+c-a-1)\Gamma_q(1+z)} \sum_{n=0}^{\infty} \frac{(q^{b-a})_n (q^{c-a})_n (q^z)_n}{(q)_n (q^{b+c-a-1})_n (q^{1+z})_n} q^{an} \\ + (1-q)^{1-z} \frac{\Gamma_q(1-a)\Gamma_q(2+a+z-b-c)}{\Gamma_q(1-b)\Gamma_q(1+z-c)\Gamma_q(1+z)} \sum_{n=0}^{\infty} \frac{(q^{b-a})_n (q^{1+a+z-c})_n}{(q^{1+z-c})_n (q^{1+z})_n} (q^{1-b})^n, \quad (3.6)$$

where

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty} (1-q)^{1-x}, \quad 0 < q < 1.$$

is the q -analogue of gamma function.

which for $q \rightarrow 1$ in (3.6) gives

$$\frac{\Gamma(z)\Gamma(b)\Gamma(c)\Gamma(b-a-z)\Gamma(c-a-z)\Gamma(1-a)}{\Gamma(a+z)\Gamma(1-a-z)\Gamma(b-a)\Gamma(c-a)\Gamma(b+c-a-z-1)} \\ = -1 + \frac{\Gamma(z)\Gamma(b)\Gamma(c)}{\Gamma(a)\Gamma(1+z)\Gamma(b+c-a-1)} \sum_{n=0}^{\infty} \frac{(b-a)_n (c-a)_n (z)_n}{n! (b+c-a-1)_n (1+z)_n}. \quad (3.7)$$

Taking $c = 1$ in (3.7) and after simplification, we obtain

$$\frac{\Gamma(z)\Gamma(b)}{\Gamma(a+z)\Gamma(b-a)} = -1 + \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{(-a)_n}{n!(z+n)}. \quad (3.8)$$

References

- [1] S. Ramanujan, Notebooks, Tata Institute of Fundamental Research, Bombay, 1957.
- [2] G. H. Hardy, Ramanujan, 3rd ed. Chelsea, New York, 1978.
- [3] S. O. Warnaar, Ramanujan's ${}_1\psi_1$ Summation, Notices of the AMS, 60, Number 1 (2013), 18-22.
- [4] C. Adiga, N. Anitha and T. Kim, Transformations of Ramanujan's Summation Formula and its applications, International J. Pure and App. Mathematical Sci., Vol.3 No.1, (2006), pp.83-91.
- [5] N. Anitha, On some Transformations of Ramanujan's ${}_1\psi_1$ Summation Formula and its applications, South East Asian J. Math. and Math. Sc., Vol.3 No.3 (2005), pp.71-81.
- [6] S. Ahmad Ali and Aditya Agnihotri, On Applications of Ramanujan's Sum, J. Math. Comput. Sci., Vol. 6, Issue 2, (2016), 156-164.
- [7] D. D. Somashekara, K. Narasimha Murthy, and S. L. Shalini, On a New Summation Formula for ${}_2\psi_2$ Basic Bilateral Hypergeometric Series and Its Applications, Int. J. of Math. And Mathematical Sciences. Vol. 2011.
- [8] G. Gasper and M. Rahman, Basic Hypergeometric Series, Second Edition. Cambridge University Press, Cambridge, 2004.
- [9] W. Y. C. Chen, Z.G. Liu, Mathematical Essays in honor of Gian-Carlo Rota, Ch. Parameter augmentation for basic hypergeometric series, I, Birkhauser, Basel (1998), 111-129.
- [10] W. Y.C. Chen, Z.G. Liu, Parameter augmentation for basic hypergeometric series, II, J. Combin. Theory. Series A 80(1997), 175-195.
- [11] Zhi-Guo Liu, Some operator identities and q-series transformation formulas, Discrete Math. 265 (2003), 119-139.
- [12] Zhizheng Zhang, Jun Wang Two operator identities and their applications to terminating basic hypergeometric series and q-integrals, J. Math. Anal. Appl. 312(2005), 653-665.